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Natural Delta Gamma hedging of longevity and interest rate risk*

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Abstract

The paper presents closed-form Delta and Gamma hedges for annuities and death assurances, in the presence of both longevity and interest-rate risk. Longevity risk is modelled through an extension of the classical Gompertz law, while interest rate risk is modelled via an Hull-and-White process. We theoretically provide natural hedging strategies, considering also contracts written on different generations. We provide a UK-population and bond-market calibrated example. We compute longevity exposures and explicitly calculate Delta-Gamma hedges. Re-insurance is needed in order to set-up portfolios which are Delta-Gamma neutral to both longevity and interest-rate risk.

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1 Introduction

Longevity risk is by now perceived as an important threat to the safety of insurance and re-insurance companies, as well as pension funds. Most actors in the financial market are long longevity risk. This makes the creation of a market for longevity a challenging task. Market makers and researchers are cooperating in transforming financial contracts subject to longevity risk in an asset class, by deepening the understanding of longevity, studying its pricing, hedging and diversification properties with respect to other asset classes.

Our contribution consists in examining life insurance liabilities subject to both longevity and interest rate risk through a parsimonious, continuous-time model for the mortality intensity - which extends the classical Gompertz law - and a benchmark model for interest rate risk. We place ourselves in a continuous-time hedging framework suggested by Cairns et al. (2008), Cairns et al. (2006a) and perform hedging in the spirit of the discrete-time coverages of Cairns (2011). In a companion paper (Luciano et al. (2011)), adopting this modelling choice, we obtained closed form formulas for the fair price of basic insurance products, i.e. pure endowments. We also obtained analytical measures of their possible changes. We considered Delta and Gamma hedge of longevity risk, as represented by the difference between the mortality intensity forecasted today and its actual realization in the future. The hedge was obtained via longevity bonds and accompanied by financial risk hedging.

In this paper we extend the Delta and Gamma computation in closed form to annuities and death assurances and study the natural hedges between contracts and between generations. On the contract side, it is indeed wise to exploit the natural offsetting between death and life contracts before using customized financial products such as OTC-longevity bonds in order to hedge longevity risk (see also Wang et al. (2010), Gatzert and Wesker (2010)). The importance of this natural way of hedging seems to go beyond risk coverage considerations, since Cox and Lin (2007) find empirical evidence that insurers whose portfolio of policies benefits from natural hedging have a competitive advantage and charge the annuitants with lower premiums. With respect to Wang et al. (2010), we include financial risk and study second-order hedges. With respect to Gatzert and Wesker (2010), we differ in the aim of hedging policy. Also, our stylized model, while probably giving up in accuracy with respect to simulations, permits to obtain closed-form formulas. On the cohort side, we differ from most other approaches in adopting a generation-based modelling of mortality: this allows us to highlight also the effect of the presence of several generations in determining the total portfolio riskiness.

The UK-calibrated example which concludes the paper confirms how relevant longevity risk is, even with respect to financial risk. In this sense, the paper complements the results in Hari et al. (2011). They that, in the absence of financial risk, longevity risk in a portfolio of annuities is substantial. In their case up to 7-8% of the initial liabilities is needed to reduce the probability of underfunding in five years to 2.5%. In our case, with both longevity and financial risk, the Deltas and Gammas with respect to longevity are greater (in absolute value) than the Greeks for financial risk. The reinsurance[1] needed for coverage - or the amount of additional premiums obtained by issuing natural-hedging policies - depends on the existence of longevity or longevity and financial risk, as well as on the number of generations in the portfolio. This comes from the fact that - with respect to Hari et al. (2011) - we have both multiple risk sources and different longevity for different generations.

The paper is structured as follows: Section 2 reviews the simplest mortality model introduced by Luciano et al. (2011), presents the corresponding formulas for the survival probability, the dynamics of the fairly-priced reserves for a pure endowment and their exposure to longevity and rate risk. It also recall the essential of the very well known Hull and White model for interest rates, which we use. Section 3 extends the pricing and hedging to annuities - considered as sequences of pure endowments - and death assurances, in order to deal with more realistic and offsetting contracts. It focuses on hedging in the presence of both contracts, i.e. natural hedging, generation by generation. Section 4 is devoted to a fully-calibrated application. Section 4.1 explains how we calibrated the model to UK data and presents the generation-based hedge ratios (Deltas and Gammas) for the case at hand. Section 4.2 computes the single-generation portfolio mix of annuities and death assurances which immunizes the portfolio up to the first and second order, comparing with the related literature. Section 4.3 studies cross-generation immunization. Section 5 concludes.

2 Longevity and interest rate risk

This section introduces the models for longevity and interest rate risk and the Delta-Gamma hedging for a pure endowment as in Luciano et al. (2011). This is intended to be only a brief review: for more details the interested reader is referred to the mentioned paper.

[1] We do not model mortality derivatives explicitly in what follows. As a consequence, we have reinsurance instead of securitization. The reader however can reinterpret re-insurance in terms of derivatives issuance. For example, Cox and Lin (2007) illustrate how mortality swaps can be used in achieving natural hedging.

2.1 Model for longevity risk

We follow a well-established stream of literature (as summarized for instance in Cairns et al. (2006a)) and we consider the time of death as the first jump time of a Poisson process with stochastic intensity, i.e. a Cox process. Let us introduce a filtered probability space $(\Omega, \mathbf{I}, \mathbb{P})$, equipped with a filtration $\{\mathcal{F}_t : 0 \leq t \leq T\}$ which satisfies the usual properties of right-continuity and completeness[2]. Our approach is generation-based. We denote with $\lambda_{Gx}(t)$ the spot instantaneous intensity at calendar time t of a head belonging to a cohort (generation) of individuals whose age is x at 0, the evaluation time. We assume that - under the $\mathbb{P}$ measure - the dynamics of the stochastic mortality intensity follows an Ornstein-Uhlenbeck process without mean reversion (OU):

$$d\lambda_{Gx}(t) = a\lambda_{Gx}(t)dt + \sigma dW_{Gx}(t)$$

where $a > 0$, $\sigma \geq 0$, W_{Gx} is a standard one-dimensional Brownian motion. Our choice of the OU process is motivated by its parsimony - very few parameters for calibration - and its appropriateness to fit human cohort-based life-tables, due to its non-mean reverting nature. It is an affine process - for which we can find closed-form expressions for the survival probability. Above all, it represents a natural stochastic generalization of the Gompertz model for the force of mortality and is thus easy to interpret in the light of the traditional actuarial practice. Its major drawback is that λ_{Gx} can turn negative with positive probability, with the survival probability decreasing in time. However, in practical applications we verify that this probability is negligible and that the survival probability is decreasing over the duration of a human life[3].

Together with the spot intensity, we will consider the forward instantaneous intensity, denoted as $f_{Gx}(t, T)$. This is the "best forecast" at time t of the spot intensity at T, since it converges to it when the horizon of the forecast goes to zero, or $T \to t$:

$$f_{Gx}(t, t) = \lambda_{Gx}(t) \tag{1}$$

[2]This filtration reflects both the mortality and financial information. For a discussion of its relationship with the natural filtration of the mortality-intensity and interest-rate processes, see Luciano et al. (2011)

[3]See Luciano and Vigna (2008). In that paper, the authors argue that the OU model - together with other non-mean reverting affine processes - seem to meet all the criteria - stated by Cairns et al. (2006a) - that a good mortality model should meet, apart from the strict positivity of the intensity. Indeed, it is consistent with historical data; its long-term future dynamics is biologically reasonable; it is comprehensive enough to deal appropriately with pricing, valuation and hedging problems; its long-term future deviations in mortality improvements are not mean-reverting. Most importantly for the case at hand, mortality-linked derivatives can be priced using analytical methods.

Standard properties of affine processes allow us to represent the survival probability from time t to T as:

$$S_{Gx}(t,T) = \mathbb{E}\left[\exp\left(-\int_t^T \lambda_{Gx}(s)ds\right) \mid \mathcal{F}_t\right] = e^{\alpha(T-t)+\beta(T-t)\lambda_{Gx}(t)} \quad (2)$$

where $\alpha(\cdot)$ and $\beta(\cdot)$ - from which we omit the subscript Gx - solve an appropriate system of Riccati differential equations with boundary conditions $\alpha(0) = \beta(0) = 0$ and have the following expressions:

$$\alpha(t) = \frac{\sigma^2}{2a^2}t - \frac{\sigma^2}{a^3}e^{at} + \frac{\sigma^2}{4a^3}e^{2at} + \frac{3\sigma^2}{4a^3} \quad (3)$$

$$\beta(t) = \frac{1}{a}(1 - e^{at}) \quad (4)$$

However, following Jarrow and Turnbull (1994), we can write the survival probability for the OU case in a more useful way as:

$$S(t,T) = \frac{S(0,T)}{S(0,t)} \exp\left[-X(t,T)I(t) - Y(t,T)\right] \quad (5)$$

with

$$X(t,T) = \frac{\exp(a(T-t)) - 1}{a}$$

$$Y(t,T) = -\sigma^2[1 - e^{2at}]X(t,T)^2/(4a)$$

$$I(t) := \lambda(t) - f(0,t)$$

$I(t)$ - the difference between the instantaneous mortality intensity at t and its forecast at time 0 - is what we interpret as the longevity risk factor: the error in forecast which makes insurance companies and pension funds exposed to longevity.

2.2 Model for interest rate risk

While in the longevity domain we have modeled first spot intensities, then forward ones, for the financial domain we follow a well-established bulk of literature, starting from Heath et al. (1992), and model directly the instantaneous forward rate $F(t,T)$, which is the time-t forecast of the instantaneous rate that will apply at time T.

Also, we assume that no arbitrages exist and we start modelling directly under a risk-neutral measure equivalent to $\mathbb{P}$, which we call $\mathbb{Q}$. We assume that the process for the forward interest rate $F(t,T)$, defined on the probability space $(\Omega, \mathbf{I}, \mathbb{Q})$, is the well-known Hull and White (1990) model, with constant parameters:

$$dF(t,T) = -gF(t,T)dt + \Sigma e^{-g(T-t)} dW_F(t) \tag{6}$$

where g is a constant parameter and W_F is a univariate Brownian motion independent of W_x for all x.
We recall that, as a particular subcase of the forward rate, obtained when $T \to t$:, one obtains the short rate process, i.e. the spot interest rate which applies instantaneously at t, which we denote as $r(t)$:

$$F(t,t) = r(t) \tag{7}$$

Under the Hull and White choice, the discount factor from T to t

$$B(t,T) = \mathbb{E}\left[\exp\left(-\int_t^T r(s)ds\right) \mid \mathcal{F}_t\right] \tag{8}$$

can be written either in a form similar to (2) or, more effectively, as

$$B(t,T) = \frac{B(0,T)}{B(0,t)} \exp\left[-\bar{X}(t,T)K(t) - \bar{Y}(t,T)\right]$$

with

$$\bar{X}(t,T) := \frac{1 - \exp(-g(T-t))}{g}$$

$$\bar{Y}(t,T) := \frac{\Sigma^2}{4g}\left[1 - \exp(-2gt)\right] \bar{X}^2(t,T)$$

where K is the financial risk factor, measured by the difference between the time-t spot and forward rate:

$$K(t) := r(t) - F(0,t)$$

2.3 Reserves for pure endowments

In the presence of both longevity and interest rate risk, the (fairly priced) reserves of any insurance product become stochastic too. This generates the need for liability hedging. In order to compute the fair value of an insurance liability, a change of measure - on the survival probability side - is still needed. We refer the reader to Luciano et al. (2011) for a detailed treatment of the issue in the current setting. Put simply, given the absence of arbitrage in the financial market, we choose a measure $\mathbb{Q}$ which allows the mortality intensity to remain of the OU type under the changed measure. This - quite standard - choice is equivalent to fixing a risk premium for longevity of the form[4]

$$\theta_{Gx}(t) = \frac{p(t) + q(t)\lambda_{Gx}(t)}{\sigma(t, \lambda_{Gx}(t))}$$

In particular, we select $p(t) = 0$ and $q(t) = q \in \mathbb{R}$.
If we maintain the assumption of independence between longevity and financial risk also after the change of measure, we can provide expressions for the fair value of insurance liabilities.

Consider first the case of a pure endowment contract starting at time 0 and paying one unit of account if the head x is alive at time T. The fair value of such an insurance policy at time $t \geq 0$ is:

$$\begin{aligned} P(t,T) &= S_{Gx}(t,T)B(t,T) \\ &= E_{\mathbb{Q}}\left[\exp\left(-\int_t^T \lambda_{Gx}(s)ds\right)|\mathcal{F}_t\right] E_{\mathbb{Q}}\left[\exp\left(-\int_t^T r(u)du\right)|\mathcal{F}_t\right] \\ &= \frac{S(0,T)}{S(0,t)}\exp\left[-X(t,T)I(t) - Y(t,T)\right]\frac{B(0,T)}{B(0,t)}\exp\left[-\bar{X}(t,T)K(t) - \bar{Y}(t,T)\right] \end{aligned} \tag{9}$$

where the parameter a showing into X, Y has been turned into $a' = a + q$.

Assuming a single premium paid at the policy issue, (9) is also the time-t reserve for the policy, that needs to be hedged by the life office.

2.4 Delta-Gamma hedging for pure endowments

Using Ito's lemma, we obtain the dynamics of the reserve $P(t,T)$ as a function of the changes in the risk factors and the first and second-order sensitivities of survival probabilities and bond prices with respect to the risk factors:

$$\begin{aligned} dP &= BdS + PdB = B\left[\frac{\partial S}{\partial t}dt + \frac{\partial S}{\partial I}dI + \frac{1}{2}\frac{\partial^2 S}{\partial I^2}(dI)^2\right] + \\ &+ S\left[\frac{\partial B}{\partial t}dt + \frac{\partial B}{\partial K}dK + \frac{1}{2}\frac{\partial^2 B}{\partial K^2}(dK)^2\right] \end{aligned}$$

Denoting the derivatives with Greek letters, and computing them, we have the following, exttremely convenient closed form solutions:

$$\Delta^M(t,T) \doteq \frac{\partial S}{\partial I} = -S(t,T)X(t,T) \leq 0 \tag{10}$$

[4]Notice that, given the absence of a rich market for longevity bonds, there are no standard choices to apply in the choice of $\theta_{Gx}(t)$: see for instance the extensive discussion in Cairns et al. (2006b). Our choice is consistent - among others - with Dahl and Møller (2006).

$$\Gamma^M(t,T) \doteq \frac{\partial^2 S}{\partial I^2} = S(t,T)X^2(t,T) \geq 0 \qquad (11)$$

$$\Delta^F(t,T) \doteq \frac{\partial B(t,T)}{\partial K} = -B(t,T)\bar{X}(t,T) \leq 0$$

$$\Gamma^F(t,T) \doteq \frac{\partial^2 B(t,T)}{\partial K^2} = B(t,T)\bar{X}^2(t,T) \geq 0$$

which lead to the analytic expansion:

$$\begin{aligned} dP &= B\left[\frac{\partial S}{\partial t}dt - SXdI + \frac{1}{2}SX^2\left(dI\right)^2\right] + \\ &+ S\left[\frac{\partial B}{\partial t}dt - BXdK + \frac{1}{2}BX^2\left(dK\right)^2\right] \end{aligned}$$

In Luciano et al. (2011) we used these coefficients to set up Delta-Gamma hedged portfolios of pure endowments[5] using longevity bonds. In the next section we extend our approach by considering more complex insurance products and show how the natural-hedging technique fits simply into our setting and allows us to compute and set up easily portfolio hedging strategies.

3 Natural Delta-Gamma hedging of annuities and death assurances

In Sections 3.1 and 3.2 we present the procedure for Delta-Gamma hedging in the presence of two basic life insurance products, namely a life annuity and a term assurance. These products are natural candidates for the natural hedging undertaken by every life office, that is also the focus of this paper. Indeed, in Section 3.3 we show how to build natural hedging in a portfolio of different policies using the different signs of the Delta and Gamma coefficients.

3.1 Annuities

Let us consider an annuity with annual installments R issued at time 0 to an individual belonging to generation Gx. Assuming the payment of a single

[5] In principle, Delta-Gamma hedging optimality in incomplete markets, such as the mixed insurance-financial one presented here, has not been demonstrated. However, Delta-Gamma hedge is one of the possible strategies open (and widely used) in those markets, together with - for instance - variance minimization strategies.

premium at policy inception, the prospective reserve L from $t \geqslant 0$ up to horizon T is:

$$L_{Gx}(t,T) = R\sum_{u=1}^{T-t} B(t,t+u)S_{Gx}(t,t+u) \tag{12}$$

The horizon T depends on the type of annuity issued. In the case of temporary annuity for n years we have $T = n$, while in the case of a whole life annuity we have $T = \omega - x$, where ω is the usual extreme age of life tables (e.g. $\omega = 120, 130...$). When computing the change in the reserve, one simply considers that this change is the sum of the changes of the pure endowment reserves with the different maturities included in the sum.

$$\begin{aligned} dL_{Gx}(t,T) &= Rd\left[\sum_{u=1}^{T-t} B(t,t+u)S_{Gx}(t,t+u)\right] &(13)\\ &= R\sum_{u=1}^{T-t}[B(t,t+u)dS_{Gx}(t,t+u) + \\ &+ dB(t,t+u)S_{Gx}(t,t+u)] &(14)\end{aligned}$$

Hence, the change in the reserve of an annuity with unit benefit ($R = 1$)- by definition - is given by the sum of the changes of the pure endowments in which it can be decomposed. Hence, the Delta and Gamma as well can be written as sums of Deltas and Gammas of pure endowments (we omit the subscript Gx for notational convenience):

$$\Delta_L^M(t,T) = -\sum_{u=1}^{T-t} B(t,t+u)S(t,t+u)X(t,t+u)$$

$$\Gamma_L^M(t,T) = \sum_{u=1}^{T-t} B(t,t+u)S(t,t+u)[X(t,t+u)]^2$$

$$\Delta_L^F(t,T) = -\sum_{u=1}^{T-t} B(t,t+u)S(t,t+u)\bar{X}(t,t+u)$$

$$\Gamma_L^F(t,T) = \sum_{u=1}^{T-t} B(t,t+u)S(t,t+u)[\bar{X}(t,t+u)]^2$$

3.2 Death assurances

Let us consider a term assurance issued at time 0 to an individual belonging to generation Gx with maturity T and sum assured C. As before, the case of whole life is covered by setting $T = \omega - x$. For simplicity, we assume that the benefit is paid at the end of the year of death, if it occurs before time T. The reserve D at time t is

$$D_{Gx}(t,T) = \sum_{u=1}^{T-t} B(t,t+u)(S_{Gx}(t,t+u-1) - S_{Gx}(t,t+u))$$

For what concerns its dynamics, we get:

$$\begin{aligned} dD_{Gx}(t,T) &= C\left[-\sum_{u=1}^{T-t} d[B(t,t+u)S_{Gx}(t,t+u)]\right. \\ &+ \left.\sum_{u=1}^{T-t} d[B(t,t+u)S_{Gx}(t,t+u-1)]\right] \end{aligned} \tag{15}$$

Hence, the following Deltas and Gammas with respect to longevity and financial risk can be computed (we omit the subscript Gx for notational convenience):

$$\begin{aligned} \Delta_D^M(t,T) &= \sum_{u=1}^{T-t} B(t,t+u)(\Delta^M(t,t+u-1) - \Delta^M(t,t+u)) \\ \Gamma_D^M &= \sum_{u=1}^{T-t} B(t,t+u)(\Gamma^M(t,t+u-1) - \Gamma^M(t,t+u)) \\ \Delta_D^F &= \sum_{u=1}^{T-t} [S(t,t+u-1) - S(t,t+u)]\, \Delta^F(t,t+u) \\ \Gamma_D^F &= \sum_{u=1}^{T-t} [S(t,t+u-1) - S(t,t+u)]\, \Gamma^F(t,t+u) \end{aligned}$$

3.3 Natural hedging within generations

Consistently with intuition, a natural offsetting is possible between the first term in (14) and (15), which represent the change in the reserve of an annuity and a death assurance respectively. Hence, even in this stochastic context, where risk comes from interest rate and longevity, the longevity exposure of an annuity provider can be reduced through positions on standard term death assurances. More than that, it is possible to quantify the exposure in closed form and to compute - still in closed form - the number of offsetting contracts.[6] Imagine for example the case of an insurer who has issued an annuity and a death assurance with the same maturity and written on the same age. Hence, the change in the value of its liabilities can be written as (we omit the subscript Gx for notational convenience):

$$\begin{aligned}
dH &= dL + dD = \sum_{u=1}^{T-t} B(t,t+u)dS(t,t+u) + dB(t,t+u)S(t,t+u)] + \\
&- \sum_{u=1}^{T-t} d[B(t,t+u)S(t,t+u)] + \sum_{u=1}^{T-t} d[B(t,t+u)S(t,t+u-1)] \\
&= \sum_{u=1}^{T-t} d[B(t,t+u)S(t,t+u-1)]
\end{aligned}$$

and it is possible to compute its Delta and Gamma by summation of the sensitivities with respect to each single liability. This is because the risk factor is the same and what differs is the exposure to the changes of the risk factor up to a certain maturity. Following this line of reasoning, we can construct – as we will do in a practical application in the next section – portfolios constituted by a mix of annuities (long exposure to longevity risk) and death assurances which are neutral with respect to first and second order changes to longevity and interest-rate risk. This is simply achieved by solving linear systems of equations in which the portfolio Greeks are set to zero. For example, consider an insurer who has issued n annuities with maturity T. He can hedge this position by setting up a (self-financed) portfolio mix of other h instruments (annuities, death assurances, bonds) which makes the total portfolio Delta-Gamma neutral with respect to both longevity and financial risk. The hedges n_i can be computed as solutions to the following system of equations, in which denote with the subscript i the dependence on the i-th hedging instrument:

$$\begin{cases}
-nL(t,T) = \sum_{i=1}^{h} n_i P_i(t,T_i) \\
-n\Delta_L^M(t,T) = \sum_{i=1}^{h} n_i \Delta_i^M(t,T_i) \\
-n\Gamma_L^M(t,T) = \sum_{i=1}^{h} n_i \Gamma_i^M(t,T_i) \\
-n\Delta_L^F(t,T) = \sum_{i=1}^{h} n_i \Delta_i^F(t,T_i) \\
-n\Gamma_L^F(t,T) = \sum_{i=1}^{h} n_i \Gamma_i^F(t,T_i).
\end{cases}$$

[6] In this sense our paper differs from Gatzert and Wesker (2010), since we have a simplified mortality setting, but closed form hedges.

We interpret negative positions as needs for policy selling and positive positions as need to buy reinsurance for exposures of that type and maturity. As a special case, let us consider for a moment longevity risk only, in the presence of a deterministic interest rate which - without loss of generality - is set to 0. We show that the natural offsetting properties of a portfolio mix of annuities and death assurances becomes more evident. The expression of the reserve associated to a death assurance simplifies remarkably:

$$D(t,T) = \sum_{u=1}^{T-t} (S(t,t+u-1) - S(t,t+u)) = 1 - S(t,T)$$

As a consequence, the sensitivities of Delta and Gamma coefficients of this contract depend only on the Delta and Gamma of a pure endowment with maturity T. Coupling an annuity and a death assurance with the same maturities in this case results in canceling out the exposure to the survival probability with maturity equal to the maturity of the death assurance (consider $C = R = 1$):

$$D(t,T) + L(t,T) = \sum_{u=1}^{T-t} (S(t,t+u) + 1 - S(t,T) = \sum_{u=0}^{T-t-1} S(t,t+u).$$

4 A calibrated example

In this section we work on a calibrated application, which concerns two generations. In Section 4.1 we explain how we calibrate the model to UK data and present the hedge ratios for the case at hand. In Section 4.2 we compute the portfolio mix of annuities and death assurances which - within each single generation - immunizes the portfolio up to the first and second order, comparing with the related literature. In section 4.3 we study cross-generation immunization, assuming, for the time being, that their longevity risks (their mortality intensities, to be precise) are perfectly, positively correlated.

4.1 Calibration to UK data

In calibrating the model to UK data we do the further - methodologically irrelevant - assumption that, for each generation, $q = 0$, so that $a' = a$, and the risk premium on longevity risk is null.[7] We therefore calibrate the

[7] This assumption could be easily removed by calibrating the model parameters to actual insurance products, whenever a liquid market for mortality derivatives will exist (see also Biffis (2005), Cairns et al. (2006b)).

Table 1: Reserves and Greeks for annuities and death assurances

Maturity	Whole-life annuity	10-year DA	20-year DA
Generation 1935 (Gy)			
Price	12.66	15.53	33.59
Δ_i^M	-269.54	1240.69	2181.05
Γ_i^M	16164.35	-20053.31	-107139.46
Δ_i^F	-95.82	-83.22	-308.34
Γ_i^F	1007.17	537.53	3406.96
Generation 1945 (Gx)			
Price	13.09	12.94	30.05
Δ_i^M	-323.48	1355.29	2619.28
Γ_i^M	24847.66	-23225.97	-146827.81
Δ_i^F	-100.92	-70.48	-285.16
Γ_i^F	1075.37	459.63	3211.46

mortality parameters to historical data, the IML tables, that are projected tables for English annuitants. We consider contracts written on the lives of male individuals who were 65 years old on 31/12/2010 (generation 1945, to which we refer as Gx), and on the lives of males who were 75 years old on 31/12/2010 (generation 1935, to which we refer as Gy) and we calibrate the OU model to those two generations. The values of the parameters, considering $t = 0$, are: $a_{Gx} = 10.94\%$, $\sigma_{Gx} = 0.07\%$, $\lambda_{Gx}(0) = -\ln p_{Gx} = 0.885\%$ for the first and $a_{Gy} = 9.95\%$, $\sigma_{Gy} = 0.03\%$, $\lambda_{Gy}(0) = -\ln p_{Gy} = 1.14\%$.[8]

In the applications to follow, we will assume first a null interest rate and consider longevity risk only; then we will introduce interest rate risk too. In order to be ready to introduce interest-rate risk, we calibrate a constant-parameter Hull-White model to the UK government bond markets at 31/12/2010 (date of reserves evaluation). The calibrated parameters for the forward-rate dynamics are $g = 2.72\%$ and $\Sigma = 0.65\%$.

For each generation, we compute the Deltas and Gammas for annuities and death assurances with different maturities. Table 1 summarizes the result for the two generations we presented above in terms of reserves and Greeks for three different products: a whole-life annuity with unit benefit and two death assurance (to which we refer from now on as DA) contracts with different maturities and insured sum equal to 100.

[8] We refer the reader to Luciano and Vigna (2008) for a full description of the data set and the calibration procedure. Notice that the calibrated parameters satisfy the sufficient condition for biological reasonableness for the OU model.

It is evident from the table that - within each single generation - the Deltas and Gammas with respect to longevity are greater (in absolute value) than the Greeks for financial risk. This happens consistently across products (for annuities and death assurances of different maturities). When, instead of looking at each single generation, we compare across them, the comparative magnitude of the greeks depends on the product. For annuities, both the longevity and financial greeks are greater (in absolute value) for the younger generation, Gx. For death assurance, the evidence is mixed.

4.2 Intra-generational naturally hedged portfolio mix

Consider now an insurer who has issued a whole-life annuity with unit benefit on an individual. Using insurance contracts and bonds, he aims at achieving instantaneous neutrality to first and second order shocks to longevity and interest rates. We compute the hedging coefficients, i.e. the positions the insurer has to hold, when he wants to cover his position using DAs with different maturities on the same generation of the annuitant.[9] Our approach is similar to the portfolio immunization framework proposed - for the insured population as a whole - by Wang et al. (2010). While they focused on natural Delta-hedging of a portfolio of annuities and term assurances, we account for both longevity and financial risk and hedge up to the second order. Imagine first the case in which the annuitant belongs to generation Gx and DAs with benefit $C = 100$ on the same cohort can be used as hedging instruments, in order to cover the reserves with an horizon T. We set ω=110. If we consider longevity risk only, by considering a deterministic interest rate equal to 0, the insurer can obtain a Delta-Hedged portfolio by issuing - for example - 0.23 DA contracts with maturity 20 years. The strategy will provide the insurer with additional liquidity – due to the initial premiums he receives – for 30.4473. He can also get a Delta-Gamma hedged portfolio by issuing 0.49 DAs with maturity 20 years and taking a long position of 0.66 on a DA with maturity 10 years (i.e. buying reinsurance on these contracts) and getting 33.2576 for having set up this strategy. In both cases, we can find a self-financing hedging strategy by adding an hedging instrument to the picture. Let us consider now also financial risk. Static Delta-Gamma hedging strategies can again be easily computed exactly, provided that we use a larger number of instruments. The insurer can cover the change in his reserves up

[9] In order to obtain a unique solution to the system of equations that solves the Delta or Delta-Gamma hedging problem, the matrix of the coefficients must be full-rank and the number of hedging instruments must equal this rank. This imposes a restriction on how many life-insurance liabilities (and bonds, when they are admitted) are used for coverage. If one accepts multiple solutions, the restriction can be relaxed.

to the first order when he has issued an annuity by issuing 0.73 DA with maturity 10 years and buying reinsurance on 0.83 DAs with maturity 20 years, getting 17.29 as a result of this strategy. A Delta-Gamma hedged portfolio involves instead taking positions on 4 DAs with different maturities, which become 5 in the case of a self-financing strategy. For example, a self-financing strategy is: 8.08 in 20 years DAs, 81.13 in 10 year DAs, -47.529 in 15 year DAs, 2.05 in 30 years DAs, -60.98 in 5 years DAs. If we make use also of bonds from the interest-rate market a self-financing Delta-Gamma hedging strategy of an annuity can be put in place by issuing 5.02 15-year term DAs, buying reinsurance for 1.50 20-year DAs and 4.68 10-year DAs and by taking positions -9.14 and 32.73 on UK-government zero-coupon bonds with 5 and 10 years maturity respectively. The only case in which a portfolio of policies issued by the insurer is naturally hedged without resorting to reinsurance is when we consider the Delta-Hedging of longevity risk only, as in Wang et al. (2010). When we consider exposure to both sources of risk, a portfolio which presents only negative positions in policies, instead, can not reach full Delta or Delta-Gamma hedging. Hence, it is sufficient to add financial risk to the picture to make it impossible to immunize the value of the liabilities to first-order shocks with the only use of standard life insurance contracts (annuities and DAs). Anyway, positions on such contracts which generate an instantaneously Delta-Gamma hedged portfolio are very easy to compute.

4.3 Hedging across generations

We now consider the case in which the insurer has a portfolio constituted by products issued on the two generations Gx and Gy. The financial risk factor obviously does not change across cohorts: hence, we can simply compute the Greeks regarding interest-rate risk by summing up the Deltas and Gammas of each liability. We have to be more careful when treating the sensitivities to the longevity risk factor. Let us assume there is a correlation, captured by a coefficient ρ between the two Gaussian risk factors $I_{Gx} = \lambda_{Gx}(t) - f_{Gx}(0, t)$ and $I_{Gy} = \lambda_{Gy}(t) - f_{Gy}(0, t)$. When this correlation coefficient is set to $\rho = 1$, we recognize that the risk factors move together.[10] Under this assumption, we can also compute portfolio sensitivities to longevity risk by simply summing up the Deltas and Gammas of each policy written on each of the two generations. We can then construct Delta-Gamma neutral portfolios by taking positions on contracts written on individuals from both cohorts.
As an example, imagine an annuity provider who has issued two annuities, one on a head of cohort Gx and the other on a head of cohort Gy. A self-

[10] The case of perfect negative correlation, $\rho = -1$ is evidently symmetric.

financing Delta-hedged portfolio to both longevity and financial risk can be set up through natural hedging by issuing 12.61 10-year DAs on an individual of cohort Gy and by buying reinsurance on 12.51 10-year DAs contracts written on cohort Gx. Importantly, if $\rho = 1$ one can achieve perfect static hedging on a product written on a certain cohort using only instruments written on other cohorts. For example, considering longevity risk only and setting a constant interest rate equal to 0, in order to Delta-hedge an annuity issued to an individual of generation Gx, the insurer can issue either 0.23(0.57) 20 (10)-year DA contracts on that generation or 0.28(0.62) 20(10)-year DA contracts on generation Gy. In both cases, setting up the strategy will provide the insurer with an additional liquidity: 30.44 (28.03) and 34.30 (30.67) respectively.

5 Conclusions

In this paper we have integrated natural hedging with the Delta-Gamma hedging technique, extending previous work by Luciano et al. (2011). We provided closed-form expressions for the Delta-Gamma hedges of annuity and death assurances subject to both longevity and financial risk. We assumed a continuous-time cohort-based model for longevity risk which generalizes the classical Gompertz law and a standard stochastic interest-rate model. We highlighted that a portfolio mix of annuities and death assurances provides natural hedging opportunities to cover longevity risk.

Our numerical application to a UK sample achieves three goals. First, it permits to compare financial and longevity sensitivities (the Greeks), intra and across generations. Second, it shows how to perform static natural hedging up to the second order in practice. Third, it allows to discuss intra and inter-generational hedge. We assume that the insurer does not want to use bonds for coverage. In this case, at least for the inter-generation case, natural hedging involving only issued policies can be achieved exclusively when Delta-hedging the portfolio against longevity risk. When we consider also financial risk or second-order effects, it is impossible to achieve perfect Delta-Gamma neutrality without making use of reinsurance transactions. Since re-insurance, in our simplified setting, summarized any resale of risk to the market, the last consideration stresses the importance of developing the mortality contracts market and of turning longevity into an asset class.

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