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Working Paper no. 29/2006
September 2006

APPLIED MATHEMATICS
WORKING PAPER SERIES

Distributions of functionals of the two parameter Poisson–Dirichlet process

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Abstract. The present paper provides exact expressions for the probability distribution of linear functionals of the two–parameter Poisson–Dirichlet process $\mathrm{PD}(\alpha, \theta)$. Distributional results that follow from the application of an inversion formula for a (generalized) Cauchy–Stieltjes transform are achieved. Moreover, several interesting integral identities are obtained by exploiting a correspondence between the mean functional of a Poisson–Dirichlet process and the mean functional of a suitable Dirichlet process. Finally, some distributional characterizations in terms of mixture representations are illustrated. Our formulae are relevant to occupation time phenomena connected with Brownian motion and more general Bessel processes, as well as to models arising in Bayesian nonparametric statistics.

Key words: α–stable subordinator; Cauchy–Stieltjes transform; Cifarelli–Regazzini identity; Functionals of random probability measures; Occupation times; Two parameter Poisson-Dirichlet process.

AMS 2000 subject classifications. Primary: 60G57. Secondary: 62F15, 60E07, 60G51

1. Introduction. Let $(P_i)_{i\geq 1}$, with $P_1 > P_2 > \ldots > 0$ and $\sum_{k=1}^{\infty} P_k = 1$, denote a sequence of (random) ranked probabilities having the two–parameter (α, θ) Poisson-Dirichlet law, denoted as $\mathrm{PD}(\alpha, \theta)$ for $0 \leq \alpha < 1$ and $\theta \geq 0$. A description, as well as a thorough investigation on its properties, is provided in [36]. See also [28], [30] and [33].

Equivalently, letting V_k, for any $k \geq 1$, denote independent random variables such that V_k has $\textsc{beta}(1-\alpha, \theta + k\alpha)$ distribution, the $\mathrm{PD}(\alpha, \theta)$ law is defined as the ranked values of the stick–breaking sequence $W_1 = V_1$, $W_k = V_k \prod_{j=1}^{k-1}(1 - V_j)$ for $k \geq 2$. Interestingly $\mathrm{PD}(\alpha, \theta)$ laws can also be obtained by manipulating random probabilities of the type $P_i = J_i/\tilde{T}$, where $\tilde{T} = \sum_{i=1}^{\infty} J_i$ and the sequence $(J_i)_{i \geq 1}$ stands for the ranked jumps of a subordinator. If the J_i's are the ranked jumps of a gamma subordinator, then the total mass $\tilde{T}$ has a gamma distribution with shape θ and scale 1 and $(P_i)_{i \geq 1}$ follows a $\mathrm{PD}(0, \theta)$ law. At the other extreme, letting the J_i's be the ranked jumps of a stable subordinator of index $0 < \alpha < 1$, $(P_i)_{i \geq 1}$ follows a $\mathrm{PD}(\alpha, 0)$ distribution. For both α and θ positive, the $\mathrm{PD}(\alpha, \theta)$ model arises by first taking the ranked jumps governed by the stable subordinator conditioned on their total mass $\tilde{T}$ and then mixing over a power tempered stable law proportional to $t^{-\theta} f_\alpha(t)$, where $f_\alpha(t)$ denotes a stable density. We also recall that there is also the case of $\mathrm{PD}(-\kappa, m\kappa)$ where $\kappa > 0$, and $m = 1, 2, \ldots$, which corresponds to symmetric Dirichlet random vectors of dimension m and parameter κ. All these models represent a natural extension of the important one–parameter family of Poisson–Dirichlet distributions, $\mathrm{PD}(0, \theta)$, which is closely connected with the Dirichlet process.

Specifically, the corresponding $\mathrm{PD}(\alpha, \theta)$ random probability measures are defined as follows. Independent of the sequence $(P_i)_{i \geq 1}$, or equivalently of the stick-breaking weights $(V_i)_{i \geq 1}$, let $(Z_i)_{i \geq 1}$ denote a collection of independent and identically distributed (iid) random elements, which take values in a Polish space $\mathbb{X}$ endowed with the Borel σ–algebra $\mathscr{X}$ and have common distribution H. Then one can construct a $\mathrm{PD}(\alpha, \theta)$ class of random probability measures, as

$$\tilde{P}_{\alpha,\theta}(\cdot) = \sum_{k=1}^{\infty} P_k \delta_{Z_k}(\cdot) = \sum_{k=1}^{\infty} W_k \delta_{Z_k}(\cdot).$$

When $\alpha = 0$ this is equivalent to the Dirichlet process which represents a cornerstone in Bayesian nonparametric statistics. See [11] and [9, 10]. The law of $\tilde{P}_{\alpha,\theta}$ may be denoted as $\mathscr{P}_{(\alpha,\theta)}(\,\cdot\,|H)$. In particular, a random probability measure with distribution $\mathscr{P}_{(-\kappa,m\kappa)}(\,\cdot\,|H)$ can be represented as

$$\tilde{P}_{-\kappa,m\kappa}(\cdot) = \sum_{i=1}^{m} \frac{G_i}{\tilde{G}}\, \delta_{Z_i}(\cdot),$$

where $\tilde{G} = \sum_{i=1}^{m} G_i$ and the G_i's are independent with $\mathrm{gamma}(\kappa, 1)$ distribution. In [31] one can find a description of this model as Fisher's model. See also [17] for more references.

The choice of $\tilde{P}_{\alpha,\theta}$ for $\alpha > 0$, or of $\tilde{P}_{-\kappa,m\kappa}$, have attractive features which make them viable models for Bayesian nonparametric analysis as shown in [31], [4] and [16, 17]. However, for the case $\alpha > 0$, most investigations about PD(α,θ) laws appear in applications related to excursion/occupation time phenomena as outlined in [36, 37] and, more recently, to combinatorial/probabilistic aspects of coalescent and phylogenetic processes. See [33] and [3] for numerous references along this line of research.

This paper studies the laws of mean functionals of the PD(α,θ) class. We also address briefly the case PD$(-\kappa, m\kappa)$, which as we shall show, essentially follows from the case of the Dirichlet process. In particular, for any non–negative valued function f such that $\tilde{P}_{\alpha,\theta}(f)$ is finite, we obtain explicit formulae for the density and the cumulative distribution function (cdf) of linear functionals

$$\tilde{P}_{\alpha,\theta}(f) = \int_{\mathbb{X}} f(x)\,\tilde{P}_{\alpha,\theta}(\mathrm{d}x) = \sum_{k=1}^{\infty} P_k f(Z_k) = \sum_{k=1}^{\infty} f(Z_k) V_k \prod_{j=1}^{k-1}(1-V_j).$$

Using a change of variable $Y_i = f(Z_i)$ with Y_i having distribution $\eta = H \circ f^{-1}$, we can equivalently work with the class of mean functionals

$$M_{\alpha,\theta}(\eta) = \sum_{k=1}^{\infty} Y_k P_k = \sum_{k=1}^{\infty} W_k Y_k.$$

As such, we extend analogous formulae for Dirichlet processes, corresponding to the case of $\alpha = 0$, given by [5]. We do this by first resorting to the Cauchy–Stieltjes transforms of order θ for $\tilde{P}_{\alpha,\theta}(g)$ models developed in [42, 43] and also to a transform of order $\theta+1$ deduced from [19], where, in particular, $\theta = 0$ for $\tilde{P}_{\alpha,0}(g)$. Then we apply an Abel-type inversion formula described in [41] and finally combine those results with mixture representations of $\tilde{P}_{\alpha,\theta}(g)$ laws. Additionally, by exploiting a correspondence between the law of $\tilde{P}_{\alpha,\theta}(g)$ and mean functionals of a Dirichlet process based on the law of $\tilde{P}_{\alpha,0}(g)$, we obtain a series of interesting and non-obvious integral identities and expectation formulae. We note that the case of $\tilde{P}_{\alpha,0}(g)$ for general g is the most tractable yielding explicit and simple expressions for the densities and cdf which are expressed in terms of Abel transforms of η, or H. The fact that our results have a strong connection to Abel transforms should not be totally surprising in view of the work in [12] where the laws of integrals of Bessel local times are investigated.

The considerations in [5], and a large body of subsequent papers, were primarily aimed at applications in Bayesian nonparametric statistics. See, e.g., [6], [39], [19], [40],[26], [15] and [27]. However it has been shown in [8], [23] and [43] that those results have implications and interpretations relevant to the Markov moment problem, continued fraction theory, exponential representations of analytic functions and so on. Since the $\mathrm{PD}(0,\theta)$ model can be seen as the limiting case of the $\mathrm{PD}(\alpha,\theta)$ distribution, as $\alpha \to 0$, we expect that some aspects of our work may be applicable to these areas as well. It is also important to note that we obtain results for pairs of the type $(\alpha, 0)$, (α,α) and $(\alpha, 1-\alpha)$. The first two are connected to lengths of excursions of Bessel processes and Bessel bridges. Moreover the important case of $\mathrm{PD}(1/2, 0)$ and $\mathrm{PD}(1/2, 1/2)$ correspond to lengths of excursions of Brownian motion and Brownian bridge, respectively. Additionally, a simple mean functional of the $\mathrm{PD}(\alpha, 1-\alpha)$ arises in the context of phylogenetic trees as recently discussed in [14]. In the next sections we describe some more details about this special case as it relates to occupation times. We then recall some results for the Dirichlet process and use this to address results for the case of $\mathrm{PD}(-\kappa, m\kappa)$. We will then devote the remainder of the paper to the study of the $\mathrm{PD}(\alpha,\theta)$ models.

1.1 Connection with occupation times for Bessel processes and models for phylogenetic trees. For functionals $\tilde{P}_{\alpha,\theta}(f)$, the generality of the space $\mathbb{X}$ is important as it allows one to formally describe phenomena, where for instance $\mathbb{X}$ denotes path spaces of stochastic processes. Surprisingly, for general (α,θ) very little is known about the laws of the simple, but important case, of $\tilde{P}_{\alpha,\theta}(f) = \tilde{P}_{\alpha,\theta}(C)$, where f coincides with the indicator function $\mathbb{I}_C$ of set $C \in \mathscr{X}$, satisfying $\mathbb{E}[\tilde{P}_{\alpha,\theta}(C)] = H(C) = p$. Hence, $f(Z) = \mathbb{I}_C(Z)$ is a Bernoulli random variable with success probability p, otherwise denoted as $\textsc{Bernoulli}(p)$. Using the stick–breaking representation, one may write

$$\tilde{P}_{\alpha,\theta}(C) = \sum_{k=1}^{\infty} Y_k V_k \prod_{j=1}^{k-1}(1-V_j)$$

where (Y_k) are iid $\textsc{Bernoulli}(p)$. The trivial case corresponds to $\tilde{P}_{0,\theta}(C)$, which is well known to be a $\textsc{Beta}(\theta p, \theta q)$ random variable, where $q = 1-p$. In fact, this is apparent from a typical construction of a Dirichlet process via its finite dimensional distributions which are Dirichlet distributed random vectors. The other known case corresponds to $\tilde{P}_{\alpha,0}(C) := Y_{\alpha,p}$, which has the Cauchy-Stieltjes transform,

$$\mathbb{E}[(1 + zY_{\alpha,p})^{-1}] = \frac{(1+z)^{\alpha-1}p + q}{(1+z)^{\alpha}p + q}. \tag{1}$$

Such a transform has been inverted in [24] yielding, as α varies in $(0,1)$, the densities

$$q_{\alpha,0}(x) = \frac{p\,q\,\sin(\alpha\pi)\,x^{\alpha-1}\,(1-x)^{\alpha-1}\,\mathbb{I}_{(0,1)}(x)}{\pi\,[q^2\,x^{2\alpha} + p^2(1-x)^{2\alpha} + 2pq\,x^{\alpha}\,(1-x)^{\alpha}\,\cos(\alpha\pi)]}\,, \tag{2}$$

otherwise known as generalized arcsine laws. It is worth noting that this, as discussed in [2], [34] and [37], also corresponds to the fraction of time spent positive by a skew Bessel process of dimension $2-2\alpha$. Precisely from [37], let $Y = (Y_t, t \geq 0)$ denote a real valued continuous process, such that (i) the zero set Z of Y is the range of a stable (α) subordinator and (ii) given $|Y|$, the signs of excursions of Y away from zero are chosen independently of each other to be positive with probability p and negative with probability $q = 1-p$. Examples of this kind of process are: Brownian motion ($\alpha = p = 1/2$); skew Brownian motion ($\alpha = 1/2$ and $0 < p < 1$); symmetrized Bessel process of dimension $2-2\alpha$; skew Bessel process of dimension $2-2\alpha$. Then for any random time T which is a measurable function of $|Y|$,

$$A_T = \int_0^T \mathbb{I}_{(0,+\infty)}(Y_s)\,\mathrm{d}s \tag{3}$$

denotes the time spent positive by Y up to time T. Furthermore remarkably $A_T/T \stackrel{d}{=} A_t/t \stackrel{d}{=} A_1 = A$ and $A \stackrel{d}{=} \tilde{P}_{\alpha,0}(C) := Y_{\alpha,p}$. We see that the case of $\alpha = 1/2$, in (2) is the density found by [22] for the fraction of time spent positive by a Brownian motion. Moreover, when $p = 1/2$ this coincides with Lévy's famous result yielding the arcsine law for Brownian motion. That is, when $p = 1/2$ the random probability $\tilde{P}_{1/2,0}(C)$ has a Beta$(1/2, 1/2)$ distribution. See [25].

In [37] it is also shown that the fraction of time spent positive by a skew Bessel bridge of dimension $2-2\alpha$ corresponds to the law of $\tilde{P}_{\alpha,\alpha}(C)$. This random variable also arises, among other places, in Corollary 33 of [32]. Another recent instance is that of $\tilde{P}_{\alpha,1-\alpha}(C)$ which equates with the limiting distribution of a phylogenetic tree model described in Proposition 20 of [14]. However, results for these models are only well known for $\alpha = 1/2$ which corresponds to skew Brownian bridges. In particular, setting $p = 1/2$ yields the Lévy result for Brownian Bridge which implies that $\tilde{P}_{1/2,1/2}(C)$ is uniform on $[0,1]$. A density for $\tilde{P}_{1/2,\theta}(C)$ and general p has been obtained by several authors, see for instance equation (3.4) in [4]. The case of $(1/2, \theta)$ when $p = 1/2$, is then Beta$(\theta + 1/2, \theta + 1/2)$. See also equation (65) in [1] for the density of $\tilde{P}_{1/2,1/2}(C)$ for general p, and yet another application related to the law of $\tilde{P}_{\alpha,\alpha}(C)$.

While the cases of Bernoulli Y_k's are indeed quite interesting we do wish to reiterate that it is substantially more difficult to obtain results for the more general case where the Y_k's have a general distribution η. In the next section we recall the results for the mean of a Dirichlet process obtained by [5] and also provide a new formula for its density.

1.2 Cifarelli and Regazzini's study of Dirichlet process mean functionals. As we noted earlier, the study of properties of Dirichlet process mean functionals has been a major area of interest in Bayesian Nonparametrics. This line of work was initiated in [5]. The authors contribution is two–fold. First they arrive at an important formula for the generalized Cauchy-Stieljtes transform of order θ of the mean functional $\tilde{P}_{0,\theta}(f)$ of the Dirichlet process $\tilde{P}_{0,\theta}$ with parameter measure $\theta\, H$. Supposing $f : \mathbb{X} \to \mathbb{R}$ is such that $\tilde{P}_{0,\theta}(|f|) < \infty$ almost surely, they show that

$$\mathbb{E}\left[\frac{1}{(1 + z\tilde{P}_{0,\theta}(f))^{\theta}}\right] = \mathrm{e}^{-\theta \int_{\mathbb{X}} \log(1+zf(y))H(\mathrm{d}y)} = \mathrm{e}^{-\theta \int_{\mathbb{R}} \log(1+zx)\eta(\mathrm{d}x)} \tag{4}$$

for any $z \in \mathbb{C}$ such that $|\arg(z)| < \pi$ and $\eta = H\circ f^{-1}$. The expression in (4) establishes that the Cauchy-Stieltjes transform of order θ of $\tilde{P}_{0,\theta}(f)$ is equivalent to the Laplace transform of $G_\theta(f)$, where $\tilde{P}_{0,\theta}(f) = G_\theta(f)/G_\theta(\mathbb{X})$, and G_θ is a gamma process with shape θH. The importance of (4) in different contexts was recognized by [8], [23] and [43]. In this regard it is called the Markov-Krein identity for means of Dirichlet processes. It is called the Cifarelli-Regazzini identity in [20]. Through no small task, [5] then apply an inversion formula to this expression to obtain an expression for the distribution of $\int x\, \tilde{P}_{0,\theta}(\mathrm{d}x)$ as follows. Let $q_{\theta\eta}$ denote the density of $\int x\, \tilde{P}_{0,\theta}(\mathrm{d}x)$, set $\Psi(x) = \int_{(0,x]} \eta(\mathrm{d}u)$ and

$$R(t) = \int_{\mathbb{R}^+\setminus\{t\}} \log(|t-x|)\, \eta(\mathrm{d}x). \tag{5}$$

Then from [5] or [6] one has for $\theta = 1$

$$q_\eta(x) = \frac{1}{\pi} \sin(\pi\Psi(x))\, \mathrm{e}^{-R(x)} \tag{6}$$

and when $\theta > 1$,

$$q_{\theta\eta}(x) = (\theta - 1) \int_0^x (x-t)^{\theta-2}\, \frac{1}{\pi}\, \sin(\pi\, \theta\, \Psi(t))\, \mathrm{e}^{-\theta R(t)}\, \mathrm{d}t. \tag{7}$$

Additionally, an expression for the cdf, which holds for $\theta\eta$ not having jumps greater than or equal to 1, is given by [5] as

$$\int_0^x (x-t)^{\theta-1}\, \frac{1}{\pi}\, \sin(\pi\, \theta\, \Psi(t))\, \mathrm{e}^{-\theta R(t)}\, \mathrm{d}t. \tag{8}$$

In particular, (8) holds for all $\theta > 0$ if η is non-atomic. We note that while there are various formulae to describe the densities of $\int x\, \tilde{P}_{0,\theta}(\mathrm{d}x)$, descriptions for the range $0 < \theta < 1$ prove to be difficult. See, e.g., [5], [39] and [26]. Here we provide a new description for the density which holds for all $\theta > 0$. This result will be obvious from our subsequent discussion concerning the inversion formula for the Cauchy–Stieltjes transform and otherwise follows immediately from (8).

PROPOSITION 1.1. *Assume that η admits a density on $\mathbb{R}^+$ and suppose R defined in* (5) *is differentiable. Then the density of the Dirichlet process mean functional $\int x\, \tilde{P}_{0,\theta}(\mathrm{d}x)$ may be expressed, for all $\theta > 0$, as*

$$q_{\theta\eta}(y) = \frac{1}{\pi}\int_0^y (y-t)^{\theta-1}\, d_{\theta,\eta}(t)\,\mathrm{d}t \tag{9}$$

where

$$d_{\theta,\eta}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\sin(\pi\,\theta\,\Psi(t))\; e^{-\theta R(t)} \tag{10}$$

It is apparent that practical usage of these formulae require tractable forms for R and of its derivative, which are not always obvious.

1.3 Laws of PD$(-\kappa, m\kappa)$ mean functionals. In this section we establish the law of the mean functional of a random probability measure with distribution $\mathscr{P}_{(-\kappa,m\kappa)}(\,\cdot\,|H)$, defined as

$$M_{-\kappa,m\kappa}(\eta) := \sum_{i=1}^m f(Z_i)\frac{G_i}{\tilde{G}} = \sum_{i=1}^m Y_i \frac{G_i}{\tilde{G}}$$

where the Y_i's are iid with common probability distribution η. One reason to study these functionals is that for the choice of $\kappa = \theta/m$ one has that $M_{-\theta/m,\theta}(\eta)$ converges in distribution to $M_{0,\theta}(\eta)$ as $m \to \infty$. This fact may be found in, e.g., [18].

It is easy to see that, conditional on $(Y_1, \ldots, Y_m)$, $M_{-\kappa,m\kappa}(\eta) \overset{d}{=} M_{0,m\kappa}(\eta_m)$, where

$$\eta_m(\,\cdot\,) = \frac{1}{m}\sum_{i=1}^m \delta_{Y_i}(\,\cdot\,) \tag{11}$$

is the empirical distribution. The Cauchy-Stieljtes transform of $M_{-\kappa,m\kappa}(\eta)$ of order $m\kappa$ is

$$\mathbb{E}\left[\frac{1}{(1+zM_{-\kappa,m\kappa}(\eta))^{m\kappa}}\right] = \left[\int_0^\infty (1+zy)^{-\kappa}\eta(\mathrm{d}y)\right]^m \qquad |\arg(z)| < \pi.$$

This leads to the following interesting result.

PROPOSITION 1.2. *The distribution of* $M_{-\kappa,m\kappa}(\eta)=\sum_{i=1}^{m}Y_i(\tilde{G}_i/\tilde{G})$, *where the* Y_i*'s are iid* η, *is described as follows. Conditional on the sequence* $(Y_i)_{i\geq 1}$, $M_{-\kappa,m\kappa}(\eta)\overset{d}{=}M_{0,m\kappa}(\eta_m)$ *where* η_m *is the empirical measure defined in* (11). *Thus descriptions of its conditional distribution, given* $(Y_i)_{i\geq 1}$, *follow from* (6), (7) *and* (8) *for appropriate ranges of the parameter* $\theta=m\kappa$, η *replaced by* η_m *and*

$$\omega_m(t)=\mathrm{e}^{-R(t)}=\prod_{i\in A_{t,m}}|t-y_i|^{-1}$$

where $A_{t,m}=\{i:\, y_i\neq t\}\cap\{1,\ldots,m\}$. *Suppose now that* $c_{m\kappa}(t):=\int_0^\infty|t-y|^{-m\kappa}\eta(\mathrm{d}y)<\infty$ *for almost every* t *with respect to the Lebesgue measure. Then one can define* $p_{m\kappa}(t)=\int_0^t(t-y)^{-m\kappa}\eta(\mathrm{d}y)/c_{m\kappa}(t)$, *and the following results hold,*

(i) *The quantity,* $\mathbb{E}_\eta[\sin(\pi m\kappa\eta_m(t))[\omega_m(t)]^{m\kappa}]$, *taking the expectation with respect to the distribution of* $(Y_i)_{i\geq 1}$, *is equal to*

$$h_{m,m\kappa}(t):=[c_{m\kappa}(t)]^m\sum_{j=1}^{m}\sin(\pi j\kappa)\binom{m}{j}[p_{m\kappa}(t)]^j[1-p_{m\kappa}(t)]^{m-j}$$

(ii) *When* $m\kappa=1$, *the density of* $M_{-1/m,1}(\eta)$ *is given by* $h_{m,1}(x)/\pi$.

(iii) *For* $\kappa=\theta/m<1$, *the cdf of* $M_{-\theta/m,\theta}(\eta)$ *is*

$$\int_0^x(x-t)^{\theta-1}\frac{1}{\pi}h_{m,\theta}(t)dt.$$

Furthermore this cdf converges to (8) *as* $m\to\infty$, *for all* $\theta>0$.

PROOF. Statement (i) is about the evaluation of $\mathbb{E}_\eta[\sin(\pi m\kappa\eta_m(t))[\omega_m(t)]^{m\kappa}]$. Here we use the fact that if $c_{m\kappa}(t)<\infty$, there exists, by a change of measure, a density for each Y_k which is proportional to $|t-y|^{-m\kappa}\eta(dy)$. It then follows that, with respect to this iid law for (Y_k), $m\eta_m(t)$ is a BINOMIAL $(mp_{m\kappa}(t))$ random variable and the result is proved. Statement (ii) is derived from (6) using a conditioning argument. Similarly statement (iii) follows from (8) noting that the jumps of $\theta\eta_m$ are less than 1. $\square$

2. An inversion formula. The present section briefly describes one of the main tools that will be used throughout the paper, i.e. the inversion formula for the (generalized)

Cauchy–Stieltjes transform of order $\theta > 0$. Some useful notation will be introduced as well.

Let $f : \mathbb{X} \to \mathbb{R}^+$ be any function in the set

$$\mathscr{E}_\alpha(H) := \left\{ f : \mathbb{X} \to \mathbb{R}^+ \quad \text{s.t.} \quad H(f^\alpha) = \int_{\mathbb{X}} f^\alpha(x)\, H(\mathrm{d}x) < +\infty \right\} \tag{12}$$

and let $\tilde{P}_{\alpha,\theta}$ denote a random probability measure with law $\mathscr{P}_{(\alpha,\theta)}(\,\cdot\,|H)$. The reason we introduce the set $\mathscr{E}_\alpha(H)$ is due to the fact that the integrability condition $\int_{\mathbb{X}} f^\alpha(x)\, H(\mathrm{d}x) < +\infty$ is necessary and sufficient for the (almost sure) finiteness of $\tilde{P}_{\alpha,0}(f)$. See Proposition 1 in [40] for a proof of this result. Hence, one can use the absolute continuity of $\mathscr{P}_{(\alpha,\theta)}(\,\cdot\,|H)$ with respect to $\mathscr{P}_{(\alpha,0)}(\,\cdot\,|H)$ in order to obtain $\tilde{P}_{\alpha,\theta}(f) < \infty$ with probability 1. For any f in $\mathscr{E}_\alpha(H)$, the transform of order $\theta > 0$ of $\tilde{P}_{\alpha,\theta}(f)$ is, for any $z \in \mathbb{C}$ such that $|\arg(z)| < \pi$,

$$\mathcal{S}_\theta\left[z; \tilde{P}_{\alpha,\theta}(f)\right] = \mathbb{E}\left[\frac{1}{(z + \tilde{P}_{\alpha,\theta}(f))^\theta}\right] = \left\{\int_{\mathbb{X}} [z + f(x)]^\alpha\ H(\mathrm{d}x)\right\}^{-\frac{\theta}{\alpha}}. \tag{13}$$

Such a representation was derived in [42] and, by a simpler proof, in [43]. This transform turns out to work well in the case where $\theta > 1$. Additionally we will need the transform of order $\theta + 1$ i.e.

$$\mathcal{S}_{\theta+1}\left[z; \tilde{P}_{\alpha,\theta}(f)\right] = \frac{\int_{\mathbb{X}} [z + f(x)]^{\alpha-1}\ H(\mathrm{d}x)}{\left\{\int_{\mathbb{X}} [z + f(x)]^\alpha\, H(\mathrm{d}x)\right\}^{\frac{\theta}{\alpha}+1}} \tag{14}$$

In particular for $\theta = 0$, we have importantly the Cauchy-Stieltjes transform of order 1 of the PD$(\alpha, 0)$ mean functionals,

$$\mathcal{S}_1\left[z;\ \tilde{P}_{\alpha,0}(f)\right] = \frac{\int_{\mathbb{X}} [z + f(x)]^{\alpha-1}\ H(\mathrm{d}x)}{\int_{\mathbb{X}} [z + f(x)]^\alpha\ H(\mathrm{d}x)} \tag{15}$$

The transforms (14) and (15) can be obtained as special cases of Proposition 6.2 in [19] with $n = 1$. Moreover, for $\theta > 0$, (14) can be obtained by taking a derivative of (13). The particular inversion formula we are going to use has been recently given in [41]. See also [7] for references on inversion formulae for generalized Cauchy–Stieltjes transforms. At this point, we anticipate a result to be given in Section 5 which establish that the probability distribution of $\tilde{P}_{\alpha,\theta}(f)$ coincides with the probability distribution of the mean of a Dirichlet process with a suitable parameter measure. Such a finding allows us to deduce that the distribution of $\tilde{P}_{\alpha,\theta}(f)$ is absolutely continuous with respect to the Lebesgue measure. See, e.g., Proposition 2 in [26]. In order to determine the density function, say $q_{\alpha,\theta}$, of $\tilde{P}_{\alpha,\theta}(f)$ we can invert $\mathcal{S}_\theta[z;\tilde{P}_{\alpha,\theta}(f)]$ as follows

$$q_{\alpha,\theta}(y) = -\frac{y^\theta}{2\pi \mathrm{i}} \int_{\mathcal{W}} (1+w)^{\theta-1}\, \mathcal{S}'_\theta[yw;\tilde{P}_{\alpha,\theta}(f)]\, \mathrm{d}w.$$

In the previous formula, $\mathcal{W}$ is a contour in the complex plane starting at -1, encircling the origin and ending at -1, while $\mathcal{S}'_\theta[yw;\tilde{P}_{\alpha,\theta}(f)] = \frac{\mathrm{d}}{\mathrm{d}z}\,\mathcal{S}_\theta[z;\tilde{P}_{\alpha,\theta}(f)]\,|_{z=yw}$. If $\theta > 1$ one can integrate by parts thus obtaining

$$q_{\alpha,\theta}(y) = \frac{\theta-1}{2\pi \mathrm{i}}\, y^{\theta-1} \int_{\mathcal{W}} (1+w)^{\theta-2} \left\{ \int_{\mathbb{X}} [yw + f(x)]^\alpha\, H(\mathrm{d}x) \right\}^{-\frac{\theta}{\alpha}} \mathrm{d}w. \tag{16}$$

Other useful formulae are obtained by considering the quantity

$$\Delta_{\alpha,\theta}(t) := \frac{1}{2\pi \mathrm{i}} \lim_{\epsilon \downarrow 0} \left\{ \left[\int_{\mathbb{X}} (-t - \mathrm{i}\epsilon + f(x))^\alpha\, H(\mathrm{d}x) \right]^{-\frac{\theta}{\alpha}} - \left[\int_{\mathbb{X}} (-t + \mathrm{i}\epsilon + f(x))^\alpha\, H(\mathrm{d}x) \right]^{-\frac{\theta}{\alpha}} \right\}.$$

Hence, one has the alternative expressions

$$q_{\alpha,\theta}(y) = \int_0^y (y-t)^{\theta-1}\, \Delta'_{\alpha,\theta}(t)\, \mathrm{d}t \tag{17}$$

and in the case $\theta > 1$ the expression above can be rewritten as follows

$$q_{\alpha,\theta}(y) = (\theta-1) \int_0^y (y-t)^{\theta-2}\, \Delta_{\alpha,\theta}(t)\, \mathrm{d}t. \tag{18}$$

See (18) and (19) in [41]. Obviously, equations (17) and (18) become useful in those cases in which $\Delta_{\alpha,\theta}$ renders finite the integrals above. Finally, note that if $\theta = 1$, then $q = \Delta_{\alpha,1}$, thus yielding the same result as in [44]. The case corresponding to $\theta < 1$ can also be dealt with by computing the transform $\mathcal{S}_{\theta+1}[z;\tilde{P}_{\alpha,\theta}(f)]$. One can, then, apply the inversion formula (18) to obtain

$$q_{\alpha,\theta}(y) = \theta \int_0^y (y-t)^{\theta-1}\, \tilde{\Delta}_{\alpha,\theta+1}(t)\, \mathrm{d}t \tag{19}$$

and

$$\tilde{\Delta}_{\alpha,\theta+1}(t) := \frac{1}{2\pi \mathrm{i}} \lim_{\epsilon \downarrow 0} \left\{ \frac{\int_{\mathbb{X}} [-t - \mathrm{i}\epsilon + f(x)]^{\alpha-1}\, H(\mathrm{d}x)}{\left[\int_{\mathbb{X}} (-t - \mathrm{i}\epsilon + f(x))^\alpha\, H(\mathrm{d}x)\right]^{\frac{\theta}{\alpha}+1}} - \frac{\int_{\mathbb{X}} [-t + \mathrm{i}\epsilon + f(x)]^{\alpha-1}\, H(\mathrm{d}x)}{\left[\int_{\mathbb{X}} (-t + \mathrm{i}\epsilon + f(x))^\alpha\, H(\mathrm{d}x)\right]^{\frac{\theta}{\alpha}+1}} \right\} \tag{20}$$

Note that the formulas (17) and (19) lead to the almost everywhere equality

$$\Delta'_{\alpha,\theta}(t) = \theta\tilde{\Delta}_{\alpha,\theta+1}(t) \tag{21}$$

for $\theta > 0$. Finally, note that $\tilde{\Delta}_{\alpha,1}(t)$ is, by Widder's inversion, the density of $\tilde{P}_{\alpha,0}(f)$. Hence, a first approach for the determination of the distribution of $\tilde{P}_{\alpha,\theta}(f)$ will aim at the determination of $\Delta_{\alpha,\theta}$ and $\tilde{\Delta}_{\alpha,\theta+1}$.

2.1 Obtaining an expression for the the cdf of $\tilde{P}_{\alpha,\theta}(f)$. Notice that once we obtain an explicit form for $\Delta_{\alpha,\theta}$, the cdf of $\tilde{P}_{\alpha,\theta}(f)$, denoted as $F_{\alpha,\theta}$ is given by

$$F_{\alpha,\theta}(x) = \int_0^x q_{\alpha,\theta}(y)\,\mathrm{d}y = \int_0^x (x-t)^{\theta-1}\Delta_{\alpha,\theta}(t)\,\mathrm{d}t \tag{22}$$

for all $\theta > 0$. This result follows by using the representation in (17) and applying integration by parts. As we shall see this representation plays a key role in obtaining simplified expressions, and obtaining various identities, for $\Delta_{\alpha,\theta}$. Hence simplifying the formulas for the densities.

2.2 Some useful notation. In this paragraph we will introduce some notations which will be used throughout the paper.

$$\mathscr{A}^+_{\eta,d}(t) = \int_t^\infty (x-t)^d\,\eta(\mathrm{d}x) \qquad \text{and} \qquad \mathscr{A}_{\eta,d}(t) = \int_0^t (t-x)^d\,\eta(\mathrm{d}x)$$

which represent generalized Abel transforms with respect to the measure η. Now define

$$\gamma_\alpha(t) = \cos(\alpha\pi)\mathscr{A}_{\eta,\alpha}(t) + \mathscr{A}^+_{\eta,\alpha}(t),$$

$$\zeta_\alpha(t) = \sin(\alpha\pi)\mathscr{A}_{\eta,\alpha}(t), \qquad \rho_{\alpha,\theta}(t) = \frac{\theta}{\alpha}\,\arctan\frac{\zeta_\alpha(t)}{\gamma_\alpha(t)} + \frac{\pi\,\theta}{\alpha}\,\mathbb{I}_\Gamma(t)$$

where $\Gamma := \{t \in \mathbb{R}^+ : \gamma_\alpha(t) < 0\}$. Note further that when $\alpha \le 1/2$, $\gamma_\alpha(t) > 0$ for all t. Moreover, define

$$\vartheta_\alpha(t) = [\gamma_\alpha(t)]^2 + [\zeta_\alpha(t)]^2 = [\mathscr{A}^+_{\eta,\alpha}(t)]^2 + 2\cos(\alpha\pi)\mathscr{A}^+_{\eta,\alpha}(t)\mathscr{A}_{\eta,\alpha}(t) + [\mathscr{A}_{\eta,\alpha}(t)]^2$$

We also use the fact, which will be proved later, that

$$\Delta_{\alpha,\alpha}(t) = \frac{1}{\pi}\frac{\zeta_\alpha(t)}{\vartheta_\alpha(t)}$$

An important thing to note at this point is that $\tilde{P}_{\alpha,\theta}(f) \stackrel{d}{=} \int x\,\tilde{P}^*_{\alpha,\theta}(\mathrm{d}x)$, where both $\tilde{P}_{\alpha,\theta}$ and $\tilde{P}^*_{\alpha,\theta}$ are Poisson-Dirichlet processes with $\mathbb{E}[\tilde{P}_{\alpha,\theta}(\,\cdot\,)] = H(\,\cdot\,)$ and $\mathbb{E}[\tilde{P}^*_{\alpha,\theta}(\,\cdot\,)] = H \circ$

$f^{-1}(\,\cdot\,) =: \eta(\,\cdot\,)$. Hence, with no loss of generality, we can confine ourselves to considering the random quantity $M_{\alpha,\theta}(\eta) := \int x\,\tilde{P}_{\alpha,\theta}(\mathrm{d}x)$. See [40] for this line of reasoning. Finally, in the following sections we will always assume f to be a function in $\mathscr{E}_\alpha(H)$.

3. Results for Means of $PD(\alpha,0)$ and generalized Arcsine Laws. We first deal with linear functionals of the PD$(\alpha,0)$ process. As recalled in the last paragraph of the previous section, with no loss of generality we can focus our attention on the random mean $M_{\alpha,0}(\eta)$. As such we generalize the results of [24] for the case of $P_{\alpha,0}(C)$. We also point out that [40] obtain an expression for the cdf of $M_{\alpha,0}(\eta)$ by exploiting a suitable inversion formula for characteristic functions and and additionally provide expressions for its posterior density. Here, the approach we exploit leads to explicit and quite tractable expressions for the density which is expressed in terms of Abel transforms of η. Moreover, we also derive new expressions for the cdf which can indeed be seen as generalized Arcsine laws.

THEOREM 3.1. *Let η be a probability measure on $(\mathbb{X},\mathscr{X})$, with $\mathbb{X}\subset\mathbb{R}^+$, and set $Q_\alpha := \{t\in\mathbb{R}^+ :\ \int_{\mathbb{X}}|x-t|^{\alpha-1}\,\eta(\mathrm{d}x) < +\infty\}$. If $\int x^\alpha\,\eta(\mathrm{d}x) < +\infty$ and the Lebesgue measure of Q_α^c is zero, then a density function of the random variable $M_{\alpha,0}(\eta) = \int x\,\tilde{P}_{\alpha,0}(\mathrm{d}x)$, denoted by $q_{\alpha,0}$ coincides with*

$$q_{\alpha,0}(t) = \frac{\sin(\alpha\pi)}{\pi}\,\frac{\mathscr{A}^+_{\eta,\alpha}(t)\mathscr{A}_{\eta,\alpha-1}(t) + \mathscr{A}^+_{\eta,\alpha-1}(t)\mathscr{A}_{\eta,\alpha}(t)}{\left[\mathscr{A}^+_{\eta,\alpha}(t)\right]^2 + 2\cos(\alpha\pi)\mathscr{A}^+_{\eta,\alpha}(t)\mathscr{A}_{\eta,\alpha}(t) + \left[\mathscr{A}_{\eta,\alpha}(t)\right]^2} \tag{23}$$

for any $t\in Q_\alpha$.

The proof is provided in Appendix A. The result for the form of the density is new. When we let α to be a parameter value lying in $(0,\frac{1}{2}]$, we are also able to obtain, in view of obvious difficulties with direct integration, a rather remarkable expression for the cumulative distribution function (cdf) given in the next theorem.

THEOREM 3.2. *Let η be a probability measure such that $\int x^\alpha\,\eta(\mathrm{d}x)$ is finite and the Lebesgue measure of the set Q_α^c is zero. Then, for any $\alpha\in(0,\frac{1}{2}]$ the cdf of $M_{\alpha,0}(\eta)$ is given by*

$$F_{\alpha,0}(t) = \frac{1}{\alpha\pi}\arctan\left(\frac{\zeta_\alpha(t)}{\gamma_\alpha(t)}\right) \tag{24}$$

Equivalently, this can be expressed in terms of a generalized arcsine distribution as follows

$$F_{\alpha,0}(t) = \frac{1}{\alpha\pi}\arcsin\left(\pi\Delta_{\alpha,\alpha}(t)[\vartheta_\alpha(t)]^{1/2}\right) \tag{25}$$

Its proof can be given along the same line followed for the proof of the next Theorem 5.2. We, then, confine ourselves to providing a proof for the latter. Specializing Theorem 3.2 to the case of $\alpha = 1/2$ we obtain the following result.

COROLLARY 3.1. *Consider the setting as in Theorem 3.1 and 3.2. Then, the density of the random variable $M_{\frac{1}{2},0}$, is given by*

$$q_{1/2,0}(t) = \frac{1}{\pi} \frac{\mathscr{A}^{+}_{\eta,1/2}(t)\mathscr{A}_{\eta,-1/2}(t) + \mathscr{A}^{+}_{\eta,-1/2}(t)\mathscr{A}_{\eta,1/2}(t)}{\left[\mathscr{A}^{+}_{\eta,1/2}(t)\right]^2 + \left[\mathscr{A}_{\eta,1/2}(t)\right]^2} \tag{26}$$

for any $t \in Q_\alpha$ and its cdf is given by the generalized arcsine distribution

$$F_{1/2,0}(t) = \frac{2}{\pi} \arcsin\left(\pi \Delta_{1/2,1/2}(t)[\vartheta_{1/2}(t)]^{1/2}\right) \tag{27}$$

REMARK 3.1. As we see the results for the PD$(\alpha, 0)$ are quite tractable and, quite remarkably, only require the calculation of the Abel transforms $\mathscr{A}_{\eta,\alpha}$ and $\mathscr{A}^{+}_{\eta,\alpha}$. In this regard one can in general obtain explicit result much more easily than for the case of the Dirchlet process. It is worth pointing out once again that our expressions for the cdf, show that these models have indeed generalized arcsine laws. These expression are rather surprising as it is not obvious how to integrate with respect to the densities.

3.1 Examples. Here below we illustrate a couple of applications of Theorem 3.1. The first one recovers a well–known result given in [24] while the second example provides an expression for the density $q_{\alpha,0}$ when η coincides with the uniform distribution on the interval $[0, 1]$.

EXAMPLE 3.1.1 (*Lamperti's occupation time density*) Here as a quick check of our results we first revisit Lamperti's model. That is to say the distribution of $\tilde{P}_{\alpha,0}(C)$. This corresponds to η being the distribution of a Bernoulli distribution with success probability $p = \mathbb{E}[\tilde{P}_{\alpha,0}(C)]$. It follows that for any $d > 0$, the Abel transforms for a Bernoulli random variable are given by,

$$\mathscr{A}^{+}_{\eta,d}(t) = (1-t)^d p \qquad \text{and} \qquad \mathscr{A}_{\eta,d}(t) = t^d(1-p) = t^d q$$

Hence, one easily sees that Lamperti's formula is recovered, i.e.

$$q_{\alpha,0}(x) = \frac{p\, q\, \sin(\alpha\pi)\, x^{\alpha-1}\, (1-x)^{\alpha-1}\, \mathbb{I}_{(0,1)}(x)}{\pi\, [q^2\, x^{2\alpha} + p^2 (1-x)^{2\alpha} + 2pq\, x^\alpha\, (1-x)^\alpha\, \cos(\alpha\pi)]}$$

and $p = 1 - q = \eta(C)$. In addition we obtain some new formula for the cdf in the case where $\alpha \le 1/2$,

$$F_{\alpha,0}(t) = \frac{1}{\alpha\pi} \arctan\left(\frac{\sin(\pi\alpha)t^\alpha q}{\cos(\alpha\pi)t^\alpha q + (1-t)^\alpha p}\right)$$

for any $t \in (0,1)$. This may also be expressed in terms of the arcsine using the fact that

$$\Delta_{\alpha,\alpha}(t) = \frac{\sin(\alpha\pi)t^\alpha q}{\pi[t^{2\alpha}q^2 + 2\cos(\alpha\pi)t^\alpha(1-t)^\alpha qp + (1-t)^{2\alpha}p^2]}$$

EXAMPLE 3.1.2 (*Uniform parameter measure*). We note again that, while there are several techniques one could have used to derive expressions for the functional $\tilde{P}_{\alpha,\theta}(C)$, it is considerably more difficult to obtain results for a more general choice of $\tilde{P}_{\alpha,\theta}(f)$, with f in $\mathscr{E}_\alpha(\eta)$. Here we demonstrate how our results easily identify the density in the case where $\eta(\mathrm{d}x) = \mathbb{I}_{(0,1)}(x)\,\mathrm{d}x$. For $M_{\alpha,0}(\eta) = \int x\,\tilde{P}_{\alpha,0}(\mathrm{d}x)$, direct calculation of the Abel transforms leads to the expression for its density as

$$q_{\alpha,0}(x) = \frac{(\alpha+1)\,\sin(\alpha\pi)\,x^\alpha\,(1-x)^\alpha}{\alpha\pi\,[x^{2\alpha+2} + (1-x)^{2\alpha+2} + 2\,\cos(\alpha\pi)\,x^{\alpha+1}\,(1-x)^{\alpha+1}]}\,\mathbb{I}_{(0,1)}(x).$$

Note that one easily finds $\gamma_\alpha(t) = (t^{\alpha+1}\cos(\alpha\pi) + (1-t)^{\alpha+1})/(\alpha+1)$ and $\zeta_\alpha(t) = t^{\alpha+1}\sin(\alpha\pi)/(\alpha+1)$, providing also an expression for the cdf. In the Dirichlet case, the distribution of $\int_{(0,1)} x\,\tilde{P}_{0,\theta}(\mathrm{d}x)$ can be determined be means of results contained in [5] and it is explicitly displayed in [8]. Its probability density function on $(0,1)$ has the form

$$q_{0,\theta}(y) = \frac{\mathrm{e}}{\pi}\,(1-y)^{-(1+y)}\,y^{-y}\,\sin(\pi y)\,\mathbb{I}_{(0,1)}(y).$$

4. $\boldsymbol{\Delta}$ formula and densities for $\boldsymbol{M_{\alpha,\theta}(\eta)}$. We are now going to consider more general cases than the α–stable process we dealt with in the previous section. As suggested by the inversion formula provided, e.g., in [41], this basically amounts to the determination of the quantities $\Delta_{\alpha,\theta}$ and $\bar{\Delta}_{\alpha,\theta+1}$. We, then, move on stating the two main results of the section and provide an interesting illustration.

THEOREM 4.1. *For any $t \in Q_\alpha$ and $(\alpha,\theta) \in (0,1) \times \mathbb{R}^+$ one has*

$$\Delta_{\alpha,\theta}(t) = \frac{1}{\pi\,[\vartheta_\alpha(t)]^{\frac{\theta}{2\alpha}}}\,\sin\left(\rho_{\alpha,\theta}(t)\right) \tag{28}$$

Additionally, one has the following alternative representation which holds for $0 < \alpha \le 1/2$,

$$\Delta_{\alpha,\theta}(t) = \frac{1}{\pi}\frac{\sin\left(\pi\theta F_{\alpha,0}(t)\right)}{[\vartheta_\alpha(t)]^{\theta/2\alpha}}$$

As far as the evaluation of $\bar{\Delta}_{\alpha,\theta+1}$ is concerned, one can prove

THEOREM 4.2. *For any $(\alpha,\theta) \in (0,1) \times \mathbb{R}^+$ and $t \in Q_\alpha$ the following equality holds true*

$$\bar{\Delta}_{\alpha,\theta+1}(t) = \frac{\gamma_{\alpha-1}(t)\,\sin(\rho_{\alpha,\theta}(t)) - \zeta_{\alpha-1}(t)\cos(\rho_{\alpha,\theta}(t))}{\pi\,[\vartheta_\alpha(t)]^{(\theta+\alpha)/(2\alpha)}} \tag{29}$$

Proofs of Theorems 4.1 and 4.2. are given in Appendix B and C, respectively. We confine ourselves to a simple illustration of the above results through an example. More detailed discussion about the determination of the probability distribution of $M_{\alpha,\theta}(\eta)$ is developed in the next sections.

EXAMPLE 4.1. (*First expressions for $\tilde{P}_{\alpha,\theta}(C)$*) It is interesting to compare the general case of $\tilde{P}_{\alpha,\theta}(C) = \tilde{P}_{\alpha,\theta}(\mathbb{I}_C)$ with that of Lamperti's result in Example 3.1. Here using the specifications in that example it follows that

$$\Delta_{\alpha,\theta}(t) = \frac{\sin\left(\frac{\theta}{\alpha}\ \arctan\left(\frac{q\ \sin(\alpha\pi)\,t^\alpha}{q\ \cos(\alpha\pi)\,t^\alpha + p\,(1-t)^\alpha}\right) + \frac{\theta}{\alpha}\,\pi\,\mathbb{I}_{\Gamma_\alpha}(t)\right)}{\pi\,\{q^2\,t^{2\alpha} + p^2\,(1-t)^{2\alpha} + 2\,q\,p\ \cos(\alpha\pi)\,t^\alpha(1-t)^\alpha\}^{\frac{\theta}{2\alpha}}} \tag{30}$$

where $\Gamma = \varnothing$ if $\alpha \in (0,1/2]$, whereas $\Gamma = (0, \frac{v_\alpha}{1+v_\alpha})$ with $v_\alpha = (-p/(q\ \cos(\alpha\pi)))^{1/\alpha}$ when $\alpha \in (1/2,1)$. From (30) one can recover the expression for the cdf of $\tilde{P}_{\alpha,\theta}(C)$ by resorting to (22). Expressions for $\tilde{\Delta}_{\alpha,\theta+1}$ can also be calculated explicitly leading to formulae for the density. In general it is evident that such results are not as amenable as the case of $\tilde{P}_{\alpha,0}(C)$, although they still lead to interesting insights. We will see that a case by case analysis can lead to more explicit expressions. We also develop other techniques in the forthcoming sections.

5. Representation of $\boldsymbol{M_{\alpha,\theta}(\eta)}$ as Dirichlet means $\boldsymbol{M_{0,\theta}(F_{\alpha,0})}$. In this section we discuss a key property which equates the law of mean functionals of the Dirichlet process with base measure corresponding to $F_{\alpha,0}$ with those of $M_{\alpha,\theta}(\eta)$ for $\theta > 0$. In principle this means that, since we have an explicit description of the $M_{\alpha,0}(\eta)$ laws, we can use the existing results of [5] to obtain expressions for the densities in the general case. However, as we noted, objects like R, defined in (5), are not easily calculated in general. In addition to the distributional relationships, basing on our results in Section 4, a series of interesting non–obvious equivalence formulae are derived.

THEOREM 5.1. *Let $\tilde{P}_{\alpha,0}$ be a normalized α–stable random measure with $\mathbb{E}[\tilde{P}_{\alpha,0}(\,\cdot\,)] = H \circ f^{-1}(\,\cdot\,) = \eta(\,\cdot\,)$. Assume further that the probability distribution of $M_{\alpha,0}(\eta)$, i.e. $F_{\alpha,0}$, is such that $\int_{\mathbb{R}^+} \log[1+x]\,\mathrm{d}F_{\alpha,0}(x) < \infty$. Then*

$$M_{\alpha,\theta}(\eta) = \tilde{P}_{\alpha,\theta}(f) \stackrel{d}{=} M_{0,\theta}(F_{\alpha,0}).$$

PROOF. From Theorem 4 in [43], note that, for any z such that $|z| < \pi$,

$$\exp\left\{-\theta \int_{\mathbb{R}^+} \log[1+zx]\,\mathrm{d}F_{\alpha,0}(x)\right\} = \left\{\int_{\mathbb{R}^+} [1+zx]^{\alpha}\,\eta(\mathrm{d}x)\right\}^{-\frac{\theta}{\alpha}}. \tag{31}$$

From the Cifarelli–Regazzini identity (4), the left-hand side in (31) coincides with the generalized Stieltjes transform of order θ of the random Dirichlet mean $M_{0,\theta}(F_{\alpha,0})$ whereas the right–hand side is the generalized Stieltjes transform of order θ of $\tilde{P}_{\alpha,\theta}(f)$. Hence the result follows. □

REMARK 5.1. We note that, although it is perhaps not immediately obvious, this fact may be deduced from a result mentioned in [35], p. 21, and attributed to [29]. This is described at the level of the laws of ranked frequencies (P_i) rather than the random probability measure but it is equivalent to that. That is, the fact that the PD(α,θ) class of models for $0 < \alpha < 1$ and $\theta > 0$ arises as a composition of the PD$(0,\theta)$ and PD$(\alpha,0)$ sequences. See also [33].

The previous Theorem 5.1 combined with the representation of the probability distribution of $M_{0,\theta}(F_{\alpha,0})$ as determined in [5] leads to an alternative representation of the key quantity $\Delta_{\alpha,\theta}$.

THEOREM 5.2. *Let $A_\alpha(t) = \int_{\mathbb{X}\setminus\{t\}} \log|t-y|\,\mathrm{d}F_{\alpha,0}(y)$. Then for all $\theta > 0$, the following results hold*

(i) $\Delta_{\alpha,\theta}(t) = \sin(\pi\theta F_{\alpha,0}(t))e^{-\theta A_\alpha(t)}$

(ii) *In particular it follows that for $t \in \{x \,:\, F_{\alpha,0}(x) > 0\}$ and $0 < \alpha \le 1/2$,*

$$e^{-A_\alpha(t)} = \left[\frac{\Delta_{\alpha,\alpha}(t)\pi}{\sin(\pi\alpha F_{\alpha,0}(t))}\right]^{1/\alpha} = [\vartheta_\alpha(t)]^{-1/2\alpha}$$

(iii) $\mathbb{E}[\log(|t - M_{\alpha,0}(\eta)|)] = A_\alpha(t) = \frac{1}{2\alpha}\log(\vartheta_\alpha(t))$ *for* $0 < \alpha \leq 1/2$.

(iv) *Statement* (ii) *implies the results* (24) *and* (25) *in Theorem 3.2.*

PROOF. Since, $M_{\alpha,\theta}(\eta) \stackrel{d}{=} M_{0,\theta}(F_{\alpha,0})$ it follows that the cdf's given in (8) and (22) are equal for all $\theta > 0$. Statement (i) then follows by the unicity properties of the integral representations. The first equivalence in statement (ii) is immediate by setting $\theta = \alpha$ in statement (i), which, noting that $0 < \alpha < 1$, uses the strict positivity $\sin(\pi\alpha F_{\alpha,0}(t))$ for $F_{\alpha,0}(t) > 0$. By similar reasoning it follows that for $\alpha \leq 1/2$

$$\mathrm{e}^{-A_\alpha(t)} = \left[\frac{\Delta_{\alpha,2\alpha}(t)\pi}{\sin(\pi 2\alpha F_{\alpha,0}(t))}\right]^{1/2\alpha} = \left[\frac{\Delta_{\alpha,\alpha}(t)\pi}{\sin(\pi\alpha F_{\alpha,0}(t))}\right]^{1/\alpha}$$

We then apply the multiple angle formula $\sin(2x) = 2\sin(x)\cos(x)$, to the expression for $\Delta_{\alpha,2\alpha}(t)$ and $\sin(\pi 2\alpha F_{\alpha,0}(t))$. Note here, we use the fact that

$$\arctan\left(\frac{\zeta_\alpha(t)}{\gamma_\alpha(t)}\right) = \arccos\left(\frac{\gamma_\alpha(t)}{[\vartheta_\alpha(t)]^{1/2}}\right) = \arcsin(\pi\Delta_{\alpha,\alpha}(t)[\vartheta_\alpha(t)]^{1/2}) \tag{32}$$

This sets up the equivalence

$$\tan(\pi\alpha F_{\alpha,0}(t)) = \frac{\zeta_\alpha(t)}{\gamma_\alpha(t)} > 0.$$

Note additionally that $0 \leq \pi\alpha F_{\alpha,0}(t) \leq \pi/2$. These points make the inverse tangent operation clear and complete the proof. □

6. Distributional results via mixture representations. In this section we describe mixture representations which are deducible from the posterior distribution of $\mathrm{PD}(\alpha,\theta)$ laws and existing results for the Dirichlet process. These represent aids in obtaining tractable forms of the distributions of various models $M_{\alpha,\theta}(\eta)$. In particular, we will use this to obtain a nice solution for all $\mathrm{PD}(\alpha, 1-\alpha)$ models. Before stating the result, let us mention in advance that $B_{a,b}$ stands for a beta distributed random variable with parameters $a >$ and $b > 0$.

THEOREM 6.1. *Let the random variables* Y, $M_{\alpha,\theta+\alpha}(\eta)$ *and* $B_{\theta+\alpha,1-\alpha}$ *be mutually independent and such that* Y *has distribution* η.

(i) *Then, for* $0 \leq \alpha < 1$, *and* $\theta \geq 0$,

$$M_{\alpha,\theta}(\eta) \stackrel{d}{=} B_{\theta+\alpha,1-\alpha}M_{\alpha,\theta+\alpha}(\eta) + (1 - B_{\theta+\alpha,1-\alpha})Y$$

Note that when $\theta > 0$ and $\alpha = 0$ this equates with the mixture representation for Dirichlet process mean functionals.

(ii) *Since $M_{\alpha,\theta}(\eta) \stackrel{d}{=} M_{0,\theta}(F_{\alpha,0})$, it follows that by setting $Y = M_{\alpha,0}(\eta)$ and $H = F_{\alpha,0}$ that for $\theta > 0$,*

$$M_{\alpha,\theta}(\eta) \stackrel{d}{=} B_{\theta,1} M_{\alpha,\theta}(\eta) + (1 - B_{\theta,1}) M_{\alpha,0}(\eta).$$

Proof. The proof follows as a direct consequence of the mixture representation of the laws of the $\tilde{P}_{\alpha,\theta}$ random probability measures deduced from their posterior distribution. Specifically one can deduce immediately from [31] with $n = 1$ that,

$$\tilde{P}_{\alpha,\theta}(\cdot) \stackrel{d}{=} B_{\theta+\alpha,1-\alpha}\, \tilde{P}_{\alpha,\theta+\alpha}(\cdot) + (1 - B_{\theta+\alpha,1-\alpha})\delta_Y(\cdot),$$

yielding the stated result. Specifically, apply the above identity to $\tilde{P}_{\alpha,\theta}(g)$, where $g(x) = x$. Naturally, this statement is an extension of the result deduced from Ferguson's characterization of a posterior distribution of a Dirichlet process. See [9, 10]. See also related discussions about mixture representations derived from posterior distributions in [19, 20]. $\square$

An immediate consequence of this result is that if one has a tractable description of the distribution of $M_{\alpha,\theta+\alpha}$, then one can easily obtain a description of the distribution of $M_{\alpha,\theta}$.

Remark 6.1. Recall that $\tilde{P}_{1/2,0}(C)$ for $p = 1/2$ has the arcsine distribution Beta$(1/2, 1/2)$. Applying the mixture representation in statement (ii) of Theorem 6.1 one can see via properties of Beta random variables that $\tilde{P}_{1/2,\theta}(C)$ is Beta$(\theta+1/2, \theta+1/2)$. This corresponds to a result of [6] for $M_{0,\theta}(\eta)$, where η is the arcsine law, although a connection to occupation time formula was not made.

Another interesting mixture representation of the distribution of the random mean $M_{\alpha,\theta}(\eta)$ arises from a combination of Theorem 5.1 with Proposition 9 in [15].

Theorem 6.2. *Let $(Q_1, \ldots, Q_k)$ denote a sequence of probability measures for $1 \leq k \leq \infty$. Additionally define $\theta_i = \theta p_i > 0$ such that $\sum_{j=1}^{k} \theta_j = \theta$. Now suppose that the cdf of $M_{\alpha,0}(\eta)$ has the mixture representation $F_{\alpha,0}(x) = \sum_{j=1}^{k} p_k Q_k(x)$. Then using the fact that $M_{\alpha,\theta}(\eta) \stackrel{d}{=} M_{0,\theta}(F_{\alpha,0})$,*

$$M_{\alpha,\theta}(\eta) \stackrel{d}{=} \sum_{i=1}^{k} D_i M_{0,\theta_i}(Q_i)$$

where $M_{0,\theta_i}(Q_i)$ for $i = 1, \ldots, k$ are independent and furthermore independent of the random vector $(D_1, \ldots, D_k)$ which is a Dirichlet $(\theta_1, \ldots, \theta_k)$ *vector. As a special case, set $Q_i = F_{\alpha,0}$ for $i = 1, \ldots, k$ then $M_{\alpha,\theta}(\eta) \stackrel{d}{=} \sum_{i=1}^k D_i M_{\alpha,\theta_i}(\eta)$.* □

Proof. From Proposition 9 in [15] one has that if the parameter measure H of a Dirichlet process admits a mixture representation $H(x) = \sum_{j=1}^k p_j\, Q_j((0,x])$, for some sequence $(Q_1, \ldots, Q_k)$ of probability measures on $\mathbb{R}^+$, then $M_{0,\theta}(H) \stackrel{d}{=} \sum_{j=1}^k D_j M_{0,\theta_j}(Q_j)$. Next, set $H = F_{\alpha,0}$ and $Q_j = F_{\alpha,0}$, for any j. Since $M_{\alpha,\theta_i}(\eta) \stackrel{d}{=} M_{0,\theta_i}(F_{\alpha,0})$, the identity in distribution easily follows. □

7. Results for PD(α, 1) and PD(α, 1 − α). We are now in a position to discuss some of the easiest and also more important cases. For example, in the case of a $PD(\alpha, 1)$ mean functional, $M_{\alpha,1}(\eta)$, it follows that its density is given by $\Delta_{\alpha,1}$ as in (28). For the range $0 < \alpha \leq 1/2$ this further reduces to

$$\Delta_{\alpha,1}(t) = \frac{1}{\pi} \sin\left(\frac{1}{\alpha} \arcsin\left(\pi \Delta_{\alpha,\alpha}(t)[\vartheta_\alpha(t)]^{1/2}\right)\right) [\vartheta_\alpha(t)]^{-1/2\alpha}. \tag{33}$$

The simplest case occurs when $\alpha = 1/2$ and, through the use of the multiple angle formula,

$$\sin(nx) = \sum_{k=0}^{n} \binom{n}{k} [\cos(x)]^k [\sin(x)]^{n-k} \sin\left(\frac{\pi}{2}[n-k]\right),$$

when $\alpha = 1/n$ where $n = 2, 3, \ldots$. We summarize these points as follows.

Theorem 7.1. *A density function of $M_{\alpha,1}(\eta)$, for all $0 < \alpha < 1$, coincides with*

$$q_{\alpha,1}(t) = \Delta_{\alpha,1}(t) = \frac{1}{\pi\ [\vartheta_\alpha(t)]^{\frac{1}{2\alpha}}}\ \sin\left(\frac{1}{\alpha}\ \arctan\frac{\zeta_\alpha(t)}{\gamma_\alpha(t)} + \frac{\pi}{\alpha}\ \mathbb{I}_\Gamma(t)\right).$$

Further simplifications arise as follows

(i) *When $\alpha = 1/2$, then the density of $M_{1/2,1}(\eta)$ is given by*

$$2\gamma_{1/2}(t)\Delta_{1/2,1/2}(t) = \frac{2}{\pi} \frac{\mathscr{A}_{1/2,\eta}(t)\mathscr{A}^+_{1/2,\eta}(t)}{\vartheta_{1/2}(t)}$$

(ii) *When $\alpha = 1/n$ for an integer $n = 3, 4, \ldots$, $M_{1/n,1}(\eta)$ has density expressible as*

$$\pi^{n-1}[\Delta_{1/n,1/n}(t)]^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{\gamma_{1/n}(t)}{\zeta_{1/n}(t)}\right)^k \sin\left(\frac{\pi}{2}[n-k]\right)$$

Now, the density of $\mathrm{PD}(\alpha, 1-\alpha)$ mean functionals can be deduced from $PD(\alpha, 1)$ models via the mixture representation given in Theorem 6.1.

THEOREM 7.2. *A density function of the random mean $M_{\alpha,1-\alpha}(\eta)$ is obtained via the distributional identity*

$$M_{\alpha,1-\alpha}(\eta) \stackrel{d}{=} B_{1,1-\alpha} M_{\alpha,1}(\eta) + (1 - B_{1,1-\alpha})Y$$

where $B_{1,1-\alpha}$, $M_{\alpha,1}(\eta)$ and Y are independent. Here Y is a random variable with distribution η. In particular the density of $M_{\alpha,1-\alpha}(\eta)$ takes the form

$$(1-\alpha) \int_0^\infty \int_0^1 \Delta_{\alpha,1}\left(\frac{x-yb}{1-b}\right)(1-b)^{-1}\, b^{-\alpha}\, \mathrm{d}b\ \eta(\mathrm{d}y)$$

7.1 The distribution of $P_{\alpha,1}(C)$ and $P_{\alpha,1-\alpha}(C)$ which is relevant to phylogenetic models. Recall from the introduction that the random variable $\tilde{P}_{\alpha,1-\alpha}(C)$, when $\mathbb{E}[\tilde{P}_{\alpha,1-\alpha}(C)] = 1/2$, is equivalent in distribution to the random variable appearing in [14]. It is known that when $\alpha = 1/2$ the distribution is uniform, according to the well–known Lèvy result. See [25]. Here we obtain a quite tractable representation of the laws for all values of α and with $\mathbb{E}[\tilde{P}_{\alpha,1-\alpha}(C)] = p$, for any $p \in (0,1)$. To this end, we first obtain the distribution of $\tilde{P}_{\alpha,1}(C)$. This can be easily obtained by setting $\theta = 1$ in (30) which yields the density function

$$f_{\alpha,1,p}(x) = \frac{\sin\left(\frac{1}{\alpha}\ \arctan\left(\frac{q\,\sin(\alpha\pi)\,t^\alpha}{q\,\cos(\alpha\pi)\,t^\alpha + p\,(1-t)^\alpha}\right) + \frac{\pi}{\alpha}\,\mathbb{I}_\Gamma(t)\right)}{\pi\,\{q^2\,t^{2\alpha} + p^2\,(1-t)^{2\alpha} + 2\,q\,p\,\cos(\alpha\pi)\,t^\alpha(1-t)^\alpha\}^{\frac{1}{2\alpha}}}$$

where $q = 1-p$, $\Gamma = \varnothing$ if $\alpha \in (0, 1/2]$, whereas $\Gamma = (0, \frac{v_\alpha}{1+v_\alpha})$ with $v_\alpha = (-p/(q\,\cos(\alpha\pi)))^{1/\alpha}$ when $\alpha \in (1/2, 1)$. Since a density function $q_{\alpha,1,p}$ of $\tilde{P}_{\alpha,1}(C)$ is available, one can evaluate $P_{\alpha,1-\alpha}(C)$, via the mixture representation stated in Theorem 7.1. It suffices to set $\eta = b_p$, where $b_p(x) = p^x(1-p)^{1-x}\,\mathbb{I}_{\{0,1\}}(x)$ is the probability mass functions of a Bernoulli random variable with parameter p. Hence, one has $\tilde{P}_{\alpha,1}(C) = M_{\alpha,1}(b_p)$ and $1 - M_{\alpha,1}(b_p) = 1 - \tilde{P}_{\alpha,1}(C) = M_{\alpha,1}(b_q)$.

COROLLARY 7.1. *Let Y denote a Bernoulli random variable with parameter p and let Y be independent of $B_{1,1-\alpha}$ and $M_{\alpha,1}(b_p)$. Then, conditional on the event $Y = 1$, one has $M_{\alpha,1-\alpha}(b_p) \stackrel{d}{=} 1 - B_{1,1-\alpha} M_{\alpha,1}(b_q)$. On the other hand, given $Y = 0$, one has*

$M_{\alpha,1-\alpha}(b_p) \stackrel{d}{=} B_{1,1-\alpha} M_{\alpha,1}(b_p)$. Equivalently a density function of $M_{\alpha,1-\alpha}(b_p)$ is obtained via the distributional relationship

$$f_{\alpha,1-\alpha,p}(t) = (1-\alpha)\int_0^1 \left[p f_{\alpha,1,p}\left(\frac{t}{u}\right) + q f_{\alpha,1,q}\left(\frac{1-t}{u}\right)\right] u^{-1}(1-u)^{-\alpha} du$$

8. The case of $\mathbf{PD}(\boldsymbol{\alpha}, \boldsymbol{\alpha})$. The important case of $\mathrm{PD}(\alpha,\alpha)$ is in general more challenging than the case of $\mathrm{PD}(\alpha, 1-\alpha)$. Of course these two agree in the case of $\alpha = 1/2$ corresponding to quantities related to Brownian bridges. Technically, one can apply the formula based on $\tilde{\Delta}_{\alpha,\alpha+1}$ but this does not always yield very nice expressions. Alternatively, in the special case where $1-\alpha = 2\alpha$, that is $\alpha = 1/3$, one might think of using mixture representation results such as those given in Theorem 6.1 and in Theorem 7.1: according to the latter one can determine $M_{\alpha,1-\alpha}(\eta)$ and, then, by resorting to the former (with $\theta = \alpha$) one obtains $M_{\alpha,\alpha}(\eta)$. Moreover, the result in Theorem 6.2 represents another useful tool. For example, one can use the Dirichlet process mixture representation to obtain the probability distribution of $M_{\frac{2}{3},\frac{2}{3}}(\eta)$ from the distribution of $M_{\frac{2}{3},\frac{1}{3}}(\eta)$. Additionally, when $\alpha > 1/2$ one may use the density representation of $M_{\alpha,2\alpha}(\eta)$ based on $\Delta_{\alpha,2\alpha}$, coupled with the mixture representation. Let us discuss about these cases by considering specific examples.

Example 8.1. (*Probability distribution of $\tilde{P}_{\alpha,\alpha}(C)$*). First note that, having set $p = \eta(C) \in (0,1)$, the following holds true

$$\Delta_{\alpha,\alpha}(t) = \frac{\sin(\alpha\pi) t^{\alpha} q}{\pi[t^{2\alpha} q^2 + 2\cos(\alpha\pi) t^{\alpha}(1-t)^{\alpha} qp + (1-t)^{2\alpha} p^2]}$$

The expressions of $\tilde{\Delta}_{\alpha,\alpha+1}$ is the same for any $\alpha \in (0,1)$ since

$$\sin\left(2\arctan\left(\frac{\zeta_\alpha(t)}{\gamma_\alpha(t)}\right)\right) = \sin\left(2\arctan\left(\frac{\zeta_\alpha(t)}{\gamma_\alpha(t)}\right) + 2\pi \mathbb{I}_\Gamma(t)\right).$$

Hence, for any $\alpha \in (0,1)$ and $t \in (0,1)$, one has

$$\tilde{\Delta}_{\alpha,\alpha+1}(t) = \frac{2\,\gamma_\alpha(t)\gamma_{\alpha-1}(t)\zeta_\alpha(t) - \zeta_{\alpha-1}(t)\gamma_\alpha^2(t) + \zeta_{\alpha-1}(t)\zeta_\alpha^2(t)}{\{\gamma_\alpha^2(t) + \zeta_\alpha^2(t)\}^2} \tag{34}$$

with $\gamma_\alpha(t) = (1-t)^\alpha + \cos(\alpha\pi)\, t^\alpha$ and $\zeta_\alpha(t) = \sin(\alpha\pi)\, t^\alpha$. These findings, with some simple algebra, lead to state the following corollary.

Corollary 8.1. *The random probability* $\tilde{P}_{\alpha,\alpha}(C)$ *admits density function coinciding with*

$$
(35)\quad q_{\alpha,\alpha}(y) = \frac{\alpha\, q\, \sin(\alpha\pi)}{\pi} \int_0^y [t(y-t)]^{\alpha-1} \times \frac{p^2(1-t)^{2\alpha-1}(1+t) + 2p\,q\,t^{\alpha+1}(1-t)^{\alpha-1}\cos(\alpha\pi) - q^2\,t^{2\alpha}}{\left[p^2(1-t)^{2\alpha} + q^2\,t^{2\alpha} + 2p\,q\,t^{\alpha}(1-t)^{\alpha}\cos(\alpha\pi)\right]^2}\,\mathrm{d}t
$$

for any y *in* $(0,1)$*, where* $q = 1-p$.

It is now worth noting that the above formula, with $\alpha = p = 1/2$, yields the well–known result about the probability distribution of A, in the case the Markov process Y is a Brownian bridge. Indeed, Lévy has found that A is uniformly distributed on the interval $(0,1)$. See [25]. In this case $\Delta_{1/2,3/2}(t) = 2\,t^{-1/2}$and the density function of $\tilde{P}_{1/2,1/2}(C)$ is given by

$$
q_{1/2,1/2}(y) = \frac{1}{2\pi}\,2\int_0^y t^{-1/2}\,(y-t)^{-1/2}\,\mathrm{d}y = 1.
$$

Example 8.2. (*Uniform parameter measure*). Let us consider again the case in which $\eta(\mathrm{d}x) = \mathbb{I}_{(0,1)}(\mathrm{d}x)$. Recall that $\gamma_\alpha(t) = (t^{\alpha+1}\,\cos(\alpha\pi) + (1-t)^{\alpha+1})/(\alpha+1)$ and $\zeta_\alpha(t) = t^{\alpha+1}\,\sin(\alpha\pi)/(\alpha+1)$. These yield

$$
\tilde{\Delta}_{\alpha,\alpha+1}(t) = \frac{\sin(\alpha\pi)\,t^{\alpha}\,\left[(1-t)^{2\alpha+1}(1+t) - t^{2\alpha+2} + 2\cos(\alpha\pi)\,t^{\alpha+2}\,(1-t)^{\alpha}\right]}{\alpha\,\left[t^{2\alpha+2} + (1-t)^{2\alpha+2} + 2\cos(\alpha\pi)\,t^{\alpha+1}(1-t)^{\alpha+1}\right]^2}
$$

The expression of the density $q_{\alpha,\alpha}$ somewhat simplifies when $\alpha = 1/2$. Indeed, in this case one has

$$
\tilde{\Delta}_{1/2,3/2}(t) = \frac{2\sqrt{t}\,[(1-t)^2(1+t) - t]}{[1-3t(1-t)]^2}
$$

for any $t \in (0,1)$. In order to determine the probability density function q, compute

$$
\begin{aligned}
I_{r,s}(y) &:= \frac{2}{\pi}\int_0^y \frac{(y-t)^{-1/2}\,t^{r+1/2}\,(1-t)^s}{[1-3t(1-t)]^2}\,\mathrm{d}t \\
&= \frac{2}{\pi}\sum_{n\ge 0}\frac{(2)_n\,3^n}{n!}\int_0^y (y-t)^{-1/2}\,t^{r+n+1/2}\,(1-t)^{n+s}\,\mathrm{d}t \\
&= \frac{2}{\pi}\sum_{n\ge 0}\frac{(2)_n\,3^n}{n!}\sum_{k=0}^{n+s}\binom{n+s}{k}(-1)^k\int_0^y (y-t)^{-1/2}\,t^{r+n+k+1/2}\,\mathrm{d}t \\
&= \frac{2}{\sqrt{\pi}}\sum_{n\ge 0}\frac{(2)_n\,3^n}{n!}\sum_{k=0}^{n+s}\binom{n+s}{k}(-1)^k\,y^{n+k+r+1}\,\frac{\Gamma(r+n+k+\frac{3}{2})}{\Gamma(r+n+k+2)}
\end{aligned}
$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ for any $a > 0$ and $n \geq 0$. Consequently

$$q_{\frac{1}{2},\frac{1}{2}}(y) = \frac{1}{\pi} \int_0^y (y-t)^{-1/2} \, \bar{\Delta}_{1/2,\,3/2}(t) \, \mathrm{d}t = I_{0,2}(y) + I_{1,2}(y) - I_{1,0}(y)$$

for any y in $(0,1)$.

An alternative representation of this density can be achieved by resorting to Theorem 7.1. Indeed one has that $M_{1/2,1/2}(\eta) \stackrel{d}{=} B_{1,1/2}\, M_{1/2,1}(\eta) + (1 - B_{1,1/2})\, Y$ where the density function of $M_{1/2,1}(\eta)$ is given by

$$q_{\frac{1}{2},1}(y) = \Delta_{\frac{1}{2},1}(y) = \frac{9}{2\,\pi} \, \frac{y^{\frac{3}{2}} \, (1-y)^{\frac{3}{2}}}{\{y^3 + (1-y)^3\}^2} \, \mathbb{I}_{(0,1)}(y)$$

and Y is uniformly distributed over the interval $(0,1)$. This, then, suggests that a density of $M_{1/2,1/2}(\eta)$ can be represented as

$$\begin{aligned} q_{\frac{1}{2},\frac{1}{2}}(y) = \frac{1}{2} \int_0^{x_1} (x_1 - x_3)^{-\frac{1}{2}} &\left\{ \int_{x_1}^1 (x_2 - x_3)^{-\frac{1}{2}} \, q_{\frac{1}{2},1}(x_2) \, \mathrm{d}x_2 \right\} \mathrm{d}x_3 \\ + \frac{1}{2} \int_{x_1}^1 (x_3 - x_1)^{-\frac{1}{2}} &\left\{ \int_0^{x_1} (x_3 - x_2)^{-\frac{1}{2}} \, q_{\frac{1}{2},1}(x_2) \, \mathrm{d}x_2 \right\} \mathrm{d}x_3 \\ = \frac{9}{2\pi} \sqrt{x_1} \int_{x_1}^1 & \frac{x_2 \, (1-x_2)^{\frac{3}{2}}}{\{x_2^3 + (1-x_2)^3\}^2} \; {}_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; \frac{x_1}{x_2}\right) \mathrm{d}x_2 \\ + \frac{9}{2\pi} \sqrt{1 - x_1} \int_0^{x_1} & \frac{x_2^{\frac{3}{2}} \, (1-x_2)}{\{x_2^3 + (1-x_2)^3\}^2} \; {}_2F_1\left(\frac{1}{2}, 1; \frac{3}{2}; \frac{1-x_1}{1-x_2}\right) \mathrm{d}x_2 \end{aligned}$$

9. Perfect Sampling $\boldsymbol{M_{\alpha,\theta}(\eta)}$. Our results so far have provided quite a few expressions for the densities and cdf's of $M_{\alpha,\theta}(\eta)$ which are certainly interesting from an analytic viewpoint. However, it is clear that if one were interested in drawing random samples it is not always obvious how to do so. The clear exception for all η is the $M_{\alpha,0}(\eta)$ case where one can apply straightforward rejection sampling based on the explicit density in Theorem 3.1. Here we show that this fact in conjunction with the correspondence to the Dirichlet process established in Theorem 5.1 and Theorem 6.1 or Theorem 6.2 allows us to perfectly sample random variables $M_{\alpha,\theta}(\eta)$ for all $0 < \alpha < 1$ and $\theta > 0$ This fact is achieved by applying the perfect sampling procedure for Dirichlet mean functionals devised by [13]. See also [21] for an application of this idea to a class of non-Gaussian Ornstein Uhlenbeck models arising in financial econometrics. Recall first that, Theorem 6.1 establishes the distributional identity

$$M_{\alpha,\eta}(\eta) \stackrel{d}{=} M_{0,\theta}(F_{\alpha,0}) \stackrel{d}{=} M_{0,\theta}(F_{\alpha,0})B_{\theta,1} + (1 - B_{\theta,1})M_{\alpha,0}(\eta).$$

Recognizing this we now recount the basic elements of the perfect sampling algorithm of [13], tailored to the present situation. First note that perfect sampling can be achieved if $0 \leq a \leq M_{\alpha,\eta}(\eta) \leq b < \infty$ almost surely. Furthermore note that this is true if and only if the support of $F_{\alpha,0}$ is $[a, b]$ or equivalently $M_{\alpha,0}(\eta) \in [a, b]$. Now as explained in [13], following the procedure of [38], one can design an upper and lower dominating chain starting at some time $-N$ in the past up to time 0. The upper chain, say $uM_{\alpha,\theta}(\eta)$, is started at $uM_{\alpha,\theta,-N}(\eta) = b$, and the lower chain, $lM_{\alpha,\theta}(\eta)$, is started at $lM_{\alpha,\theta,-N}(\eta) = a$. One runs the Markov chains for each n based on the equations,

$$uM_{\alpha,\theta,n+1}(\eta) = B_{n,\theta}X_n + (1 - B_{n,\theta})uM_{\alpha,\theta,n}(\eta)$$

and

$$lM_{\alpha,\theta,n+1}(\eta) = B_{n,\theta}X_n + (1 - B_{n,\theta})lM_{\alpha,\theta,n}(\eta)$$

where the chains are coupled using the same random independent pairs $(B_{n,\theta}, X_n)$ where for each n, $B_{n,\theta}$ has a Beta$(1, \theta)$ distribution and X_n has distribution $F_{\alpha,0}$. That is $X_n \stackrel{d}{=} M_{\alpha,0}(\eta)$. The chains are said to coalesce when $D = |uM_{\alpha,\theta,n}(\eta) - luM_{\alpha,\theta,n}(\eta)| < \epsilon$ for some small ϵ. Notice importantly that this method only requires the ability to sample $M_{\alpha,0}(\eta)$, which is provided by Theorem 3.1, and an independent Beta random variable.

APPENDIX

Proof of Theorem 3.1 The first thing to note is that

$$\mathcal{S}_1[z; M_{\alpha,0}(\eta)] = \frac{\int [z + x]^{\alpha-1}\, \eta(\mathrm{d}x)}{\int [z + x]^{\alpha}\, \eta(\mathrm{d}x)} \tag{36}$$

for any z such that $|\arg(z)| < \pi$. Now let $G_t = (0, t)$ and

$$\gamma_{\epsilon,\alpha}(t) = \int_{\mathbb{R}} \left[(x - t)^2 + \epsilon^2\right]^{\frac{\alpha}{2}} \cos\left(\alpha \arctan \frac{\epsilon}{x - t} + \alpha\, \pi\, \mathbb{I}_{G_t}(x)\right) \eta(\mathrm{d}x)$$
$$\zeta_{\epsilon,\alpha}(t) = \int_{\mathbb{R}} \left[(x - t)^2 + \epsilon^2\right]^{\frac{\alpha}{2}} \sin\left(\alpha \arctan \frac{\epsilon}{x - t} + \alpha\, \pi\, \mathbb{I}_{G_t}(x)\right) \eta(\mathrm{d}x),$$

In order to evaluate the density $q_{\alpha,\eta}$, one can invert (36) by means of the Perron–Stieltjes formula which yields

$$q_{\alpha,\eta}(y) = \frac{1}{2\pi\mathrm{i}} \lim_{\epsilon \downarrow 0} \{\mathcal{S}_1[-y - \mathrm{i}\epsilon; M_{\alpha,0}(\eta)] - \mathcal{S}_1[-y + \mathrm{i}\epsilon; M_{\alpha,0}(\eta)]\}$$

and it can be seen that the above reduces to

$$q_{\alpha,\eta}(y) = \frac{1}{\pi} \lim_{\epsilon\downarrow 0} \text{Im } \{\mathcal{S}_1[-y - \mathrm{i}\epsilon; M_{\alpha,0}(\eta)]\} = \frac{1}{\pi} \lim_{\epsilon\downarrow 0} \text{Im } \frac{\gamma_{\epsilon,\alpha-1}(y) - \mathrm{i}\,\zeta_{\epsilon,\alpha-1}(y)}{\gamma_{\epsilon,\alpha}(y) - \mathrm{i}\,\zeta_{\epsilon,\alpha}(y)}$$

The assumptions $\int_{\mathbb{R}^+} x^\alpha\,\eta(\mathrm{d}x) < \infty$ and y in Q_α allow a straightforward application of the dominated convergence theorem. This leads to $\lim_{\epsilon\downarrow 0} \gamma_{\epsilon,\alpha}(y) = \gamma_\alpha(y)$ and $\lim_{\epsilon\downarrow 0} \zeta_{\epsilon,\alpha}(y) = \zeta_\alpha(y)$ for any y, while $\lim_{\epsilon\downarrow 0} \gamma_{\epsilon,\alpha-1}(y) = \gamma_{\alpha-1}(y)$ and $\lim_{\epsilon\downarrow 0} \zeta_{\epsilon,\alpha-1}(y) = \zeta_{\alpha-1}(y)$ for any $y \in Q_\alpha$. The result, then, easily follows. □

Proof of Theorem 4.1. First note that since one has $\tilde{P}_{\alpha,\theta} = \sum_{j\geq 1} \tilde{p}_j\,\delta_{X_j}$, where the the (random) weights $\tilde{p}_j$ are non–negative, sum up to one and are independent from the locations X_j which are independent and identically distributed (i.i.d.) with common probability distribution is η. This clearly implies that the support of $M_{\alpha,\theta}(\eta) = \int x\,\tilde{P}_{\alpha,\theta}(\mathrm{d}x) = \sum_{j\geq 1} X_j\,\tilde{p}_j$ is the closure of the convex hull of the support of η, i.e. $\text{supp}(M_{\alpha,\theta}(\eta)) = \overline{\text{co}}(\text{supp}(\eta)) =: C(\eta)$. Now, from the definition of $\Delta_{\alpha,\theta}$ and from the representation of the generalized Stieltjes transform of $M_{\alpha,\theta}(\eta)$, as given in [42] and in [43], it is apparent that

$$\Delta_{\alpha,\theta}(t) = \frac{1}{\pi} \lim_{\epsilon\downarrow 0} \text{Im}\,\mathcal{S}_\theta\,[-t - \mathrm{i}\epsilon; M_{\alpha,\theta}(\eta)] = \frac{1}{\pi} \lim_{\epsilon\downarrow 0} \text{Im } \left\{\int_{\mathbb{X}} (-t - \mathrm{i}\epsilon + x)^\alpha\,\eta(\mathrm{d}x)\right\}^{-\frac{\theta}{\alpha}}$$

where $\mathbb{X} \subset \mathbb{R}^+$. One has

$$\left\{\int_{\mathbb{R}} (-t - \mathrm{i}\epsilon + x)^\alpha\,\eta(\mathrm{d}x)\right\}^{-\frac{\theta}{\alpha}} = \exp\left\{-\frac{\theta}{\alpha} \log\left(\gamma_{\epsilon,\alpha}(t) - \mathrm{i}\,\zeta_{\epsilon,\alpha}(t)\right)\right\}$$

Let us first confine our attention to the case in which α is in the interval $(0, 1/2]$. Since $\alpha \arctan(\frac{\epsilon}{x-t}) + \alpha\,\pi\,\mathbb{I}_{(0,t)}(x) \in (0, \alpha\pi)$, for any t and x, one has $\zeta_{\epsilon,\alpha}(t) > 0$ and$\gamma_{\epsilon,\alpha}(t) > 0$. Consequently,

$$\exp\left\{-\frac{\theta}{\alpha} \log\left(\gamma_{\epsilon,\alpha}(t) - \mathrm{i}\,\zeta_{\epsilon,\alpha}(t)\right)\right\} = \left\{\gamma^2_{\epsilon,\alpha}(t) + \zeta^2_{\epsilon,\alpha}(t)\right\}^{-\frac{\theta}{2\alpha}} \exp\left\{\mathrm{i}\,\frac{\theta}{\alpha} \arctan\frac{\zeta_{\epsilon,\alpha}(t)}{\gamma_{\epsilon,\alpha}(t)}\right\}.$$

Note that the absolute value of each of the two integrands defining $\gamma_{\epsilon,\alpha}$ and $\zeta_{\epsilon,\alpha}$ are bounded by $|x - t|^\alpha + K$ which is integrable with respect to η. We can, then, apply a dominated convergence argument to obtain

$$\lim_{\epsilon\downarrow 0} \gamma_{\epsilon,\alpha}(t) = \gamma_\alpha(t) \qquad \lim_{\epsilon\downarrow 0} \zeta_{\epsilon,\alpha}(t) = \zeta_\alpha(t)$$

for any $t > 0$. This implies (28) after noting that, in this case, $\Gamma = \varnothing$.

On the other hand, when $\alpha \in (1/2, 1)$, one needs to consider the set $\Gamma_\epsilon := \{t \in \mathbb{R}^+ : \gamma_{\epsilon,\alpha}(t) > 0\}$ and note that $\Gamma_\epsilon^c \cap (0, y)$ is non–empty for some values of y in $C(\eta)$. This yields a slightly different form for the arguments of the complex numbers involved in the definition of $\Delta_{\alpha,\theta}$. One can easily mimic the line of reasoning employed for the case $\alpha \in (0, 1/2]$ so to obtain, again, (28). $\square$

Proof of Theorem 4.2. By definition

$$\Delta_{\alpha,\theta+1}(t) = \frac{1}{\pi} \lim_{\epsilon \downarrow 0} \{\mathcal{S}_{\theta+1}[-t - \mathrm{i}\epsilon; M_{\alpha,\theta}(\eta)] - \mathcal{S}_{\theta+1}[-t + \mathrm{i}\epsilon; M_{\alpha,\theta}(\eta)]\}$$

which can be seen to imply

$$\Delta_{\alpha,\theta+1}(t) = \frac{1}{\pi} \lim_{\epsilon \downarrow 0} \mathrm{Im}\, \frac{\int_{\mathbb{R}} (-t - \mathrm{i}\epsilon + x)^{\alpha-1}\, \eta(\mathrm{d}x)}{\left\{\int_{\mathbb{R}} (-t - \mathrm{i}\epsilon + x)^{\alpha}\, \eta(\mathrm{d}x)\right\}^{(\theta+\alpha)/\alpha}}$$

For any $\epsilon > 0$, $|(-t - \mathrm{i}\epsilon + x)^\alpha|$ can be bounded by an integrable function, with respect to η, not depending on ϵ in a similar fashion as in the proof of Theorem 4.1. On the other hand $|(-t - \mathrm{i}\epsilon + x)^{\alpha-1}| \leq |x - t|^{\alpha-1} + K'$ for any $\epsilon > 0$ and for any x and t. If we further set $t \in Q_\alpha$, then $x \mapsto |x - t|^{\alpha-1}$ is integrable, with respect to η, and the dominated convergence theorem can be applied. The expression in (29) easily follows. $\square$

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