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Zonoids, Linear Dependence, and Size-Biased Distributions on the Simplex*

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Abstract

The zonoid of a d-dimensional random vector is used as a tool for measuring linear dependence among its components. A preorder of linear dependence is defined through inclusion of the zonoids.

The zonoid of a random vector does not characterize its distribution, but it characterizes the size biased distribution of its compositional variables. This fact will allow a characterization of our linear dependence order in terms of a linear-convex order for the size-biased compositional variables. In dimension 2 the linear dependence preorder will be shown to be weaker than the concordance order.

Some examples related to the Marshall-Olkin distribution and to a copula model will be presented, and a class of measures of linear dependence will be proposed.

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Keywords: zonoid, zonotope, linear dependence, compositional variables, multivariate size biased distribution, concordance order, Marshall-Olkin distribution.

1 Introduction

The theory of zonotopes and zonoids goes back a long time (the reader is referred to Bolker (1969), Goodey and Weil (1993), and Schneider (1993), for the theory of zonoids and extensive bibliographies on the topic). Only recently zonoids were introduced by Koshevoy and Mosler (1996, 1998) in the statistical literature with the aim of measuring multivariate inequality.

By using a simple characterization contained in Dall'Aglio and Scarsini (2001), we can interpret these zonoids as ranges of suitable nonatomic vector measures. For instance, given a nonatomic probability space $(\Omega, \mathcal{F}, P)$, the lift zonoid of a random vector $\mathbf{X} = (X_1, \ldots, X_d)$ is the range of the $(d+1)$-dimensional vector measure $(P, P_{X_1}, \ldots, P_{X_d})$ such that the Radon-Nikodym derivative $dP_{X_i}/dP = X_i$ (P-a.s.). The Lorenz zonoid is similarly defined by replacing the random variables X_i with the normalized random variables $X_i/E[X_i]$. It is not difficult to see that, for $d = 1$ the Lorenz zonoid is the area between the Lorenz curve and its dual. We refer the reader to the above-quoted papers and to Mosler (2002) for the properties and the interpretation of the different zonoids. In particular Koshevoy and Mosler (1996, 1998) showed how to construct an order for random vectors based on the inclusion of their lift zonoids. They proved that the lift zonoid of a random vector $\mathbf{X}$ is contained in the lift zonoid of another random vector $\mathbf{Y}$ if and only if every linear combination $\langle \boldsymbol{\alpha}, \mathbf{X} \rangle$ is dominated in the convex order by the corresponding linear combination $\langle \boldsymbol{\alpha}, \mathbf{Y} \rangle$. Therefore the inclusion of lift zonoids defines an order of variability for random vectors. In dimension 1, as is well known, this coincides with the usual dilation order of random variables. In higher dimension this is weaker than the dilation order.

Since the shape of the lift zonoid depends heavily on the dependence structure of the random vector, it is natural to ask whether, at least for random vectors with same marginals, we could use the inclusion of lift zonoids to define an order of dependence (in some sense to be defined). A result of Scarsini and Shaked (1990) shows that for random vectors $\mathbf{X}, \mathbf{Y}$ with the same marginals, $\langle \boldsymbol{\alpha}, \mathbf{X} \rangle$ is dominated in the convex order by $\langle \boldsymbol{\alpha}, \mathbf{Y} \rangle$ for all $\boldsymbol{\alpha} \in \mathbb{R}^d$ if and only if $\mathbf{X}$ and $\mathbf{Y}$ have the same law. This fact does not leave much hope to extricate information about the comparative dependence of $\mathbf{X}$ and $\mathbf{Y}$ by looking at the inclusion of their lift zonoids.

In this paper we will show that, for nonnegative random vectors, a simpler object, namely, the zonoid, does the job. The zonoid of $\mathbf{X}$ is the range of the d-dimensional vector measure $(P_{X_1}, \ldots, P_{X_d})$, or, in other words, the projection of the lift zonoid on its last d dimensions. Heuristically it tells us how the different components of the random vector $\mathbf{X}$ tend to spread their mass with respect to one another, but not how they behave with respect to the original measure P on $(\Omega, \mathcal{F})$. This implies that the zonoid (unlike the lift zonoid) does not characterize the law of the random vector. This fact prevents dispersion comparisons based on the zonoid, but it is the key element to allow dependence comparisons, at least for positive random vectors.

The kind of dependence that is captured by the zonoid is linear in the sense that, among all the distributions on the positive orthant with the same expectation, the zonoid is the smallest when all the components of the random vector are proportional to one another, and it is the largest when all the mass is deposited on the main axes. Therefore we can define an order of linear dependence through the inclusion of the zonoids. We will show that, for $d = 2$ this linear dependence order is implied by the concordance order.

What we do in this paper bears some analogy with the analysis of positive dependence based on the idea of copula, namely, by considering the zonoid instead of the lift zonoid we have thrown away the components of the multivariate distribution that depend on the marginals and have concentrated our attention on the dependence structure of the distribution.

In Section 2 we relate the zonoid of a nonnegative random vector to a distribution on the simplex. In Section 3 we use the above connection to define an ordering of linear dependence for nonnegative random vectors. In Section 4 we compare the above ordering with the concordance ordering. In Section 5 we apply the ordering of linear dependence to an exchangeable Marshal-Olkin class and to a copula model. Finally in Section 6 we define some measures of linear dependence that are consistent with the ordering.

The following notational conventions will be used throughout the paper. Given two points $\mathbf{s}, \mathbf{t} \in \mathbb{R}^d$, we denote by $\overline{\mathbf{s}, \mathbf{t}}$ the segment with endpoints $\mathbf{s}$ and $\mathbf{t}$, by $\langle \mathbf{s}, \mathbf{t} \rangle = \sum_{i=1}^d s_i t_i$ their inner product, and by $\mathbf{s} \vee \mathbf{t} = (\max\{s_1, t_1\}, \ldots, \max\{s_d, t_d\})$, and $\mathbf{s} \wedge \mathbf{t} = (\min\{s_1, t_1\}, \ldots, \min\{s_d, t_d\})$ their lattice operators. We use the symbols

$\mathbf{0} = (0, \dots, 0)$ and $\mathbf{1} = (1, \dots, 1)$.

Given two sets $A, B \in \mathbb{R}^d$, $A \oplus B = \{\mathbf{s}+\mathbf{t} : \mathbf{s} \in A, \mathbf{t} \in B\}$ is their Minkowski sum.

We denote by Σ_{d-1} the $(d-1)$-dimensional simplex

$$\Sigma_{d-1} = \left\{ \mathbf{t} \in \mathbb{R}^d : t_i \geq 0, \sum_{j=1}^{d} t_j = 1 \right\},$$

and by $\mathcal{S}^{d-1} = \{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{x}, \mathbf{x} \rangle = 1\}$ the unit sphere in $\mathbb{R}^d$.

The orders $\leq$ and $>$ on $\mathbb{R}^d$ are intended componentwise, namely, $\mathbf{s} \leq (>)\mathbf{t}$ iff $s_i \leq (>)t_i$ for $i = 1, \dots, d$.

Given a random vector $\mathbf{X}$ on $(\Omega, \mathcal{F}, P)$, $\mathcal{L}_P(\mathbf{X})$ denotes its law.

2 Zonoids and distributions on the simplex

Let $(\Omega, \mathcal{F}, P)$ be a nonatomic probability space. Without any loss of generality we can choose $(\Omega, \mathcal{F}, P) = ([0,1], \mathrm{Bor}([0,1]), \mathrm{Leb})$, namely, the unit interval endowed with the Borel σ-field and the Lebesgue measure.

Let $\mathcal{X}^d(P)$ be the class of d-dimensional random vectors $\mathbf{X} = (X_1, \dots, X_d)$ defined on $(\Omega, \mathcal{F}, P)$ such that

$$\int_\Omega \sum_{i=1}^{d} |X_i(\omega)| \; P(d\omega) < \infty,$$

let $\mathcal{X}_+^d(P) \subset \mathcal{X}^d(P)$ be the class of nonnegative d-dimensional random vectors $\mathbf{X}$ such that $P(\mathbf{X} = \mathbf{0}) = 0$, and let $\mathcal{X}_+^d(\boldsymbol{\mu}, P) \subset \mathcal{X}_+^d(P)$ be the class of nonnegative random vectors such that

$$\int_\Omega X_i(\omega) \; P(d\omega) = \mu_i < \infty,$$

with $\boldsymbol{\mu} = (\mu_1, \dots, \mu_d)$.

Unless otherwise indicated all the statements concerning random vectors will be understood to be under the probability measure P. When a result holds under a different measure Q, say, this will be indicated with the symbol $|_Q$.

Given a random vector $\mathbf{X} = (X_1, \dots, X_d) \in \mathcal{X}^d(P)$, define a vector measure $\mathbf{P}_{\mathbf{X}} = (P_{X_1}, \dots, P_{X_d})$ on $(\Omega, \mathcal{F})$ with values in $\mathbb{R}^d$ as follows

$$P_{X_i}(A) = \int_A X_i(\omega) \; P(d\omega), \qquad \text{for } A \in \mathcal{F}, \qquad i = 1, \dots, d. \tag{2.1}$$

By construction the measure $\mathbf{P_X}$ is nonatomic on $(\Omega, \mathcal{F})$. We indicate the range of a vector measure $\mathbf{P}$ on $\mathcal{F}$ by $\mathbf{P}(\mathcal{F})$.

Definition 2.1. We define

(a) $Z_P(\mathbf{X}) = \mathbf{P_X}(\mathcal{F})$,

(b) $\ell_P(\mathbf{X}) = (P, \mathbf{P_X})(\mathcal{F})$.

If $\mathbf{X}$ is nonnegative, then the quantities $Z_P(\mathbf{X})$ and $\ell_P(\mathbf{X})$ lie in the positive orthant, and are called respectively zonoid and lift-zonoid of $\mathbf{X}$. It not difficult to see that $\ell_P(\mathbf{X}) = Z_P(1, \mathbf{X})$.

For general properties of zonoids the reader is referred to Bolker (1969), which contains, among other things the following characterization of a zonoid. Any zonoid containing the origin is a limit in the Hausdorff metric of some sequence of zonotopes, where a zonotope is a finite sum of segments with one end in the origin.

Some of the most interesting results in the theory of zonoids go back to Blaschke and were then elegantly proved, among others, by Choquet (1969). For the sake of completeness we will report them here in the form of Schneider (1993, Thm. 3.5.2, 3.5.3), to which we refer for relevant references.

Given a convex body $K \subseteq \mathbb{R}^d$, its support function $h(K, \cdot) : \mathbb{R}^d \to \mathbb{R}$ is defined as $h(K, \mathbf{u}) = \sup\{\langle \mathbf{x}, \mathbf{u} \rangle : \mathbf{x} \in K\}$. A compact convex set is characterized by its support function.

Theorem 2.2. *A convex body $K \subseteq \mathbb{R}^d$ is a zonoid with center at $\mathbf{0}$ iff its support function can be represented in the form*

$$h(K, \mathbf{x}) = \int_{\mathcal{S}^{d-1}} |\langle \mathbf{x}, \mathbf{v} \rangle| \, d\rho(\mathbf{v}), \quad \text{for } \mathbf{x} \in \mathbb{R}^d,$$

with some even measure ρ on $\mathcal{S}^{d-1}$.

Theorem 2.3. *If ρ is a signed even measure on $\mathcal{S}^{d-1}$ with*

$$\int_{\mathcal{S}^{d-1}} |\langle \mathbf{u}, \mathbf{v} \rangle| \, d\rho(\mathbf{v}) = 0, \quad \text{for } \mathbf{u} \in \mathbb{R}^d,$$

then $\rho = 0$.

The above theorems imply that a zonoid with center at $\mathbf{0}$ is characterized by a unique even measure on the sphere $\mathcal{S}^{d-1}$. Using the above results we can associate to each nonnegative random vector $\mathbf{X} \in \mathcal{X}_+^d$ a unique distribution on the simplex Σ_{d-1}. We will provide a simple heuristics for this distribution.

Given a random vector $\mathbf{X} \in \mathcal{X}_+^d(\boldsymbol{\mu}, P)$, define a measure $Q^{\mathbf{X}}$ on $(\Omega, \mathcal{F})$ as follows: For $A \in \mathcal{F}$,

$$Q^{\mathbf{X}}(A) = \int_A \frac{\sum_{i=1}^d X_i(\omega)}{\sum_{i=1}^d \mu_i} \, P(d\omega).$$

The distribution of $\mathbf{X}$ under the measure $Q^{\mathbf{X}}$ can be seen as a multivariate size-biased version of its distribution under P.

If we define, for $i = 1, \dots, d$, the compositional variables generated by $(X_1, \dots, X_d)$

$$\widetilde{X}_i = \frac{X_i}{\sum_{j=1}^d X_j}, \tag{2.2}$$

then it is clear that the random vector $\widetilde{\mathbf{X}} := (\widetilde{X}_1, \dots, \widetilde{X}_d)$ takes values in Σ_{d-1}.

It is immediate to verify that, if $\mathbf{X} \in \mathcal{X}_+^d(\boldsymbol{\mu}, P)$, then

$$E_{Q^{\mathbf{X}}}[\widetilde{X}_i] = \frac{\mu_i}{\sum_{j=1}^d \mu_j}.$$

Theorem 2.4. *Let $\mathbf{X}, \mathbf{Y} \in \mathcal{X}_+^d(\boldsymbol{\mu}, P)$ and let $\widetilde{\mathbf{X}}, \widetilde{\mathbf{Y}}$ be defined as in* (2.2). *Then*

(a) $Z_{Q^{\mathbf{X}}}(\widetilde{\mathbf{X}}) = (\sum_{i=1}^d \mu_i)^{-1} Z_P(\mathbf{X})$.

(b) $Z_P(\mathbf{X}) = Z_P(\mathbf{Y})$ *if and only if* $\mathcal{L}_{Q^{\mathbf{X}}}(\widetilde{\mathbf{X}}) = \mathcal{L}_{Q^{\mathbf{Y}}}(\widetilde{\mathbf{Y}})$.

Theorem 2.4 is an immediate corollary of Theorems 2.2 and 2.3, due to the fact that the simplex Σ_{d-1} is homeomorphic to a part of the half-sphere in $\mathcal{S}^{d-1}$. The distribution of $\widetilde{\mathbf{X}}$ under $Q^{\mathbf{X}}$ is the projection on the simplex Σ_{d-1} of the corresponding distribution of $\mathbf{X}$. The definition of the zonoid of a random vector requires to size-bias this distribution on the simplex in order to obtain the original zonoid (modulo a multiplicative constant).

3 An order of linear dependence

Definition 3.1. The linear dependence preorder $\leq_{\text{ld}}$ on $\mathcal{X}_+^d(\boldsymbol{\mu}, P)$ is defined as follows

$$\mathbf{Y} \leq_{\text{ld}} \mathbf{X} \quad \text{if} \quad Z_P(\mathbf{X}) \subset Z_P(\mathbf{Y}). \tag{3.1}$$

Let $\mathbf{X}^+ = (X_1^+, \ldots, X_d^+) \in \mathcal{X}_+^d(\boldsymbol{\mu}, P)$ be a random vector such that

$$\frac{X_1^+}{\mu_1} = \frac{X_2^+}{\mu_2} = \cdots = \frac{X_d^+}{\mu_d}, \tag{3.2}$$

and let $\mathbf{X}^- = (X_1^-, \ldots, X_d^-) \in \mathcal{X}_+^d(\boldsymbol{\mu}, P)$ be a random vector such that, if $X_i^- > 0$, then $X_j^- = 0$ for $j \neq i$.

The vector $\mathbf{X}^+$ represents a situation of maximal positive linear dependence, in that its components are proportional to one another. The vector $\mathbf{X}^-$ represents a situation of minimal positive dependence since it concentrates all the mass on the main axes, i.e. as far as possible from the line of proportionality determined by $\overline{\mathbf{0}, \boldsymbol{\mu}}$. It is clear that $\mathbf{X}^+$ and $\mathbf{X}^-$ are not unique.

Proposition 3.2. *For every $\mathbf{X} \in \mathcal{X}_+^d(\boldsymbol{\mu}, P)$ we have*

$$\mathbf{X}^- \leq_{\mathrm{ld}} \mathbf{X} \leq_{\mathrm{ld}} \mathbf{X}^+.$$

Proof. We know that $Z_P(\mathbf{X})$ always contains the points $\mathbf{0}$ and $\boldsymbol{\mu}$ and therefore, by convexity it contains the segment $\overline{\mathbf{0}, \boldsymbol{\mu}}$. By (3.2) for all $i, j = 1, \ldots, d$, and for all $A \in \mathcal{F}$

$$\frac{1}{\mu_i} P_{X_i^+}(A) = \frac{1}{\mu_j} P_{X_j^+}(A),$$

therefore $Z_P(\mathbf{X}^+) = \overline{\mathbf{0}, \boldsymbol{\mu}}$.

The set $Z_P(\mathbf{X})$ is compact, convex, symmetric with respect to $\frac{1}{2}\boldsymbol{\mu}$, contains $\mathbf{0}$, and lies in the positive orthant, therefore it is contained in the hypercube $\times_{i=1}^d [0, \mu_i]$. Since $Z_P(\mathbf{X}^-)$ contains the points $\mu_1 \mathbf{e}_1, \ldots, \mu_d \mathbf{e}_d$ and $\boldsymbol{\mu}$, by convexity $Z_P(\mathbf{X}^-) = \times_{i=1}^d [0, \mu_i]$. □

The following stochastic orders are well known in the literature (see e.g. Shaked and Shanthikumar (1994), Scarsini (1998)). We say that $\mathbf{X}$ is dominated by $\mathbf{Y}$ in the convex order ($\mathbf{X} \leq_{\mathrm{cx}} \mathbf{Y}$) if $E\phi(\mathbf{X}) \leq E\phi(\mathbf{Y})$ for all convex functions $\phi : \mathbb{R}^d \to \mathbb{R}$ for which the expectation exists.

We say that $\mathbf{X}$ is dominated by $\mathbf{Y}$ in the linear convex order ($\mathbf{X} \leq_{\mathrm{lincx}} \mathbf{Y}$) if $E\psi(\langle \boldsymbol{\alpha}, \mathbf{X} \rangle) \leq E\psi(\langle \boldsymbol{\alpha}, \mathbf{Y} \rangle)$ for all $\boldsymbol{\alpha} \in \mathbb{R}^d$ and all convex functions $\psi : \mathbb{R} \to \mathbb{R}$ for which the expectation exists.

Theorem 3.3. *Let $\mathbf{X}, \mathbf{Y} \in \mathcal{X}_+^d(\boldsymbol{\mu}, P)$, and let $\widetilde{\mathbf{X}}, \widetilde{\mathbf{Y}}$ be defined as in* (2.2)*. Then the following two conditions are equivalent*

(i) $\mathbf{Y} \leq_{\text{ld}} \mathbf{X}$

(ii) $\widetilde{\mathbf{X}}|_{Q^{\mathbf{X}}} \leq_{\text{lincx}} \widetilde{\mathbf{Y}}|_{Q^{\mathbf{Y}}}$.

Lemma 3.4 (Theorem 5.2 of Koshevoy and Mosler (1998)). *The following two conditions are equivalent*

(a) $\mathbf{X} \leq_{\text{lincx}} \mathbf{Y}$,

(b) $\ell_P(\mathbf{X}) \subset \ell_P(\mathbf{Y})$.

Proof of Theorem 3.3.

$$\begin{aligned}\mathbf{Y} \leq_{\text{ld}} \mathbf{X} \iff Z_P(\mathbf{X}) \subset Z_P(\mathbf{Y}) \iff Z_{Q^{\mathbf{X}}}(\widetilde{\mathbf{X}}) \subset Z_{Q^{\mathbf{Y}}}(\widetilde{\mathbf{Y}}) \iff \\ \ell_{Q^{\mathbf{X}}}(\widetilde{\mathbf{X}}) \subset \ell_{Q^{\mathbf{Y}}}(\widetilde{\mathbf{Y}}) \iff \widetilde{\mathbf{X}}|_{Q^{\mathbf{X}}} \leq_{\text{lincx}} \widetilde{\mathbf{Y}}|_{Q^{\mathbf{Y}}},\end{aligned}$$

where the first implication stems from Definition 3.1, the second from Theorem 2.4, and the last one from Lemma 3.4. The third implication is implied by the fact that if a random vector $\mathbf{W}$ has values in Σ_{d-1}, then

$$\sum_{i=1}^{d} P_{W_i}(A) = P(A).$$

□

Theorem 3.5. *Let* $\mathbf{X}, \mathbf{Y} \in \mathcal{X}_+^d(\boldsymbol{\mu}, P)$*, and let* $\widetilde{\mathbf{X}}, \widetilde{\mathbf{Y}}$ *be defined as in* (2.2)*. If* $\widetilde{\mathbf{X}}|_{Q^{\mathbf{X}}} \leq_{\text{cx}} \widetilde{\mathbf{Y}}|_{Q^{\mathbf{Y}}}$*, then* $\mathbf{Y} \leq_{\text{ld}} \mathbf{X}$.

Proof. This follows immediately from the fact that $\widetilde{\mathbf{X}}|_{Q^{\mathbf{X}}} \leq_{\text{cx}} \widetilde{\mathbf{Y}}|_{Q^{\mathbf{Y}}}$ implies $\widetilde{\mathbf{X}}|_{Q^{\mathbf{X}}} \leq_{\text{lincx}} \widetilde{\mathbf{Y}}|_{Q^{\mathbf{Y}}}$. □

The converse of Theorem 3.5 does not hold in general, as the following counterexample shows.

Example 3.6. Let $d = 3$ and

$$P(\mathbf{X} = (0,0,4)) = P(\mathbf{X} = (0,4,0)) = P(\mathbf{X} = (4,0,0)) = P(\mathbf{X} = (4,4,4)) = \frac{1}{4},$$
$$P(\mathbf{Y} = (0,3,3)) = P(\mathbf{Y} = (3,0,3)) = P(\mathbf{Y} = (3,3,0)) = \frac{1}{3}.$$

It is not difficult to see that $E[\mathbf{X}] = E[\mathbf{Y}]$, and $Z(\mathbf{X}) \supset Z(\mathbf{Y})$.

The random vectors $\widetilde{\mathbf{X}}$ and $\widetilde{\mathbf{Y}}$ are distributed as follows:

$$Q^{\mathbf{X}}(\widetilde{\mathbf{X}} = (0,0,1)) = Q^{\mathbf{X}}(\widetilde{\mathbf{X}} = (0,1,0)) = Q^{\mathbf{X}}(\widetilde{\mathbf{X}} = (1,0,0)) = \frac{1}{6}$$

$$Q^{\mathbf{X}}\left(\widetilde{\mathbf{X}} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\right) = \frac{1}{2},$$

$$Q^{\mathbf{Y}}(\widetilde{\mathbf{Y}} = (0, \frac{1}{2}, \frac{1}{2})) = Q^{\mathbf{Y}}(\widetilde{\mathbf{Y}} = (\frac{1}{2}, 0, \frac{1}{2})) = Q^{\mathbf{Y}}(\widetilde{\mathbf{Y}} = (\frac{1}{2}, \frac{1}{2}, 0)) = \frac{1}{3},$$

therefore $\widetilde{\mathbf{X}}|_{Q^{\mathbf{X}}}$ is not convex-dominated by $\widetilde{\mathbf{Y}}|_{Q^{\mathbf{Y}}}$.

The converse of Theorem 3.5 holds for $d = 2$

Theorem 3.7. *Let* $\mathbf{X}, \mathbf{Y} \in \mathcal{X}_+^2(\boldsymbol{\mu}, P)$*, and let* $\widetilde{\mathbf{X}}, \widetilde{\mathbf{Y}}$ *be defined as in* (2.2)*. If* $\mathbf{Y} \leq_{\text{ld}} \mathbf{X}$*, then* $\widetilde{\mathbf{X}}|_{Q^{\mathbf{X}}} \leq_{\text{cx}} \widetilde{\mathbf{Y}}|_{Q^{\mathbf{Y}}}$.

Proof. Again the result stems from the fact that, for $d = 2$, $\widetilde{\mathbf{X}}|_{Q^{\mathbf{X}}} \leq_{\text{cx}} \widetilde{\mathbf{Y}}|_{Q^{\mathbf{Y}}}$ is equivalent to $\widetilde{\mathbf{X}}|_{Q^{\mathbf{X}}} \leq_{\text{lincx}} \widetilde{\mathbf{Y}}|_{Q^{\mathbf{Y}}}$. □

Even if the proof of Theorem 3.7 is quite trivial, we want to provide an alternative simple constructive argument, for which we need the following definitions and results from [Elton and Hill (1992)].

Definition 3.8. Given a probability measure Q on $(\mathbb{R}^d, \text{Bor}(\mathbb{R}^d))$, and a set $A \in \text{Bor}(\mathbb{R}^d)$, we say that A has finite first moment if

$$\int_A \sum_{i=1}^d |t_i| \, dQ(\mathbf{t}) < \infty.$$

If $Q(A) > 0$ and A has finite first moment, let

$$\mathbf{b}(A, Q) = \frac{1}{Q(A)} \int_A \mathbf{t} \, dQ(\mathbf{t})$$

be the Q-barycenter of A.

Definition 3.9. Given two probability measures Q', Q'' on $(\mathbb{R}^d, \text{Bor}(\mathbb{R}^d))$, we say that Q'' is an elementary fusion of Q' if there exist an $A \in \text{Bor}(\mathbb{R}^d)$ with finite first Q'-moment and a $\alpha \in [0,1]$ such that

$$Q'' = Q'\|_{A^c} + \alpha Q'(A)\delta(\mathbf{b}(A, Q')) + (1-\alpha)Q'\|_A \tag{3.3}$$

where $\delta(\mathbf{x})$ is the degenerate probability measure in $\mathbf{x}$, and $Q'\|_A$ is the restriction of the measure Q' to the set A.

We say that Q'' is a general fusion of Q' if there exist $B \in \mathrm{Bor}(\mathbb{R}^d)$, $\beta \in [0,1]$ and three probability measures Q^*, Q^{**}, Q^{***}, such that $Q^{**}(B^c) = 0$, Q^{***} is an elementary fusion of Q^{**}, and

$$Q' = \beta Q^* + (1-\beta)Q^{**},$$

and

$$Q'' = \beta Q^* + (1-\beta)Q^{***}.$$

Lemma 3.10 (Theorem 4.1 of Elton and Hill (1992)). *Given two probability measures Q', Q'' on $(\mathbb{R}^d, \mathrm{Bor}(\mathbb{R}^d))$, the following two conditions are equivalent*

(a) $Q' \leq_{\mathrm{cx}} Q''$,

(b) *there exists a sequence $\{Q_n\}$ of probability measures such that $Q_0 = Q''$, $Q_n \to_{\mathrm{weakly}} Q'$ and Q_{n+1} is an elementary fusion of Q_n.*

Since the ordering $\leq_{\mathrm{cx}}$ is an integral ordering (see Müller (1997)), and an elementary fusion is a particular case of general fusion, we have the following corollary.

Corollary 3.11. *Given two probability measures Q', Q'' on $(\mathbb{R}^d, \mathrm{Bor}(\mathbb{R}^d))$, the following two conditions are equivalent*

(a) $Q' \leq_{\mathrm{cx}} Q''$,

(b) *there exists a sequence $\{Q_n\}$ of probability measures such that $Q_0 = Q''$, $Q_n \to_{\mathrm{weakly}} Q'$ and Q_{n+1} is a general fusion of Q_n.*

We start considering the case when $Z_P(\mathbf{Y})$ is a zonotope. For $\mathbf{X}, \mathbf{Y} \in \mathcal{X}_+^2(\boldsymbol{\mu}, P)$, let $\mathbf{Y} \leq_{\mathrm{ld}} \mathbf{X}$, namely, $Z_P(\mathbf{X}) \subset Z_P(\mathbf{Y})$.

Consider a point $\mathbf{t} \in \partial Z_P(\mathbf{X})$ such that $\mathbf{t} \notin \partial Z_P(\mathbf{Y})$ and a supporting line $h_{\mathbf{t}}$ of $Z_P(\mathbf{X})$ through $\mathbf{t}$. Consider then the point $\mathbf{t}'$ symmetric to $\mathbf{t}$ with respect to $\frac{1}{2}\boldsymbol{\mu}$ and the line $h_{\mathbf{t}'}$ through $\mathbf{t}'$ parallel to $h_{\mathbf{t}}$. The two lines $h_{\mathbf{t}}$ and $h_{\mathbf{t}'}$ partition $\mathbb{R}^2$ into three regions. Call them $H_1(\mathbf{t}, \mathbf{t}')$, $H_2(\mathbf{t}, \mathbf{t}')$, and $H_3(\mathbf{t}, \mathbf{t}')$ where H_2 is the closed set between the two lines. The body $Z_1 := Z_P(\mathbf{Y}) \cap H_2(\mathbf{t}, \mathbf{t}')$ is a new zonoid such that

$$Z_P(\mathbf{X}) \subset Z_1 \subset Z_P(\mathbf{Y}).$$

The zonoid Z_1 is a zonotope and can be obtained by replacing some of the segments that generate $Z_P(\mathbf{Y})$ with one single segment. To be more precise consider the segments in $\partial Z_P(\mathbf{Y}) \cap H_3(\mathbf{t}, \mathbf{t}')$. Take segments of the same length and direction with an endpoint in the origin. These are either whole segments or parts of segments generating $Z_P(\mathbf{Y})$. Replace these segments (or part of segments) with the segment parallel to $Z_P(\mathbf{Y}) \cap h_{\mathbf{t}}$. It is not difficult to see that the effect of this replacement on the distribution of $\widetilde{\mathbf{Y}}$ is just a general fusion. Now iterate the procedure, namely choose a point $\mathbf{t}_2 \in \partial Z_P(\mathbf{X})$ such that $\mathbf{t}_2 \notin \partial Z_1$, and generate a new zonoid Z_2 as above in such a way that

$$Z_P(\mathbf{X}) \subset Z_2 \subset Z_1.$$

If the sequence of points $\mathbf{t}, \mathbf{t}_2, \ldots$ is chosen appropriately, then it will give rise to a decreasing sequence $\{Z_n\}$ of zonoids such that $Z_n \searrow Z_P(\mathbf{X})$. Now it is enough to apply Corollary 3.11 to get the result.

In order to prove the theorem for any distribution of $\mathbf{Y}$ we will have to adapt the result in Bolker (1969) showing that zonoids are limits, in the Hausdorff metric, of zonotopes. Since $Z_P(\mathbf{Y})$ is contained in the positive orthant and has $\mathbf{0}$ as vertex, we will need to approximate it with zonotopes that have the same property, namely, are contained in the positive orthant and have $\mathbf{0}$ as vertex. In $\mathbb{R}^2$ this can be done by intersecting an arbitrary approximating sequence of zonotopes with the rectangle $[0, \mu_1] \times [0, \mu_2]$. A similar construction could not be carried out in dimension $d > 2$.

The above argument is similar in spirit to some constructions used by Chacon and Walsh (1976) for the Skorohod embedding, and by Machina and Pratt (1997) for the Rothschild and Stiglitz (1970) characterization of the convex order.

The argument breaks down in dimension $d > 2$, since what we get by cutting a zonoid with two parallel hyperplanes is in general not a zonoid.

4 Linear dependence and concordance

The class of bivariate random vectors can be ordered in terms of concordance as follows: $\mathbf{X} \leq_{\text{conc}} \mathbf{Y}$ if

$$P(\{\mathbf{X} \leq \mathbf{t}\} \cup \{\mathbf{X} > \mathbf{t}\}) \leq P(\{\mathbf{Y} \leq \mathbf{t}\} \cup \{\mathbf{Y} > \mathbf{t}\}) \qquad \forall \mathbf{t} \in \mathbb{R}^2.$$

The concordance order is often called Positive Quadrant Dependence order. For properties of this order the reader is referred for instance to Kimeldorf and Sampson (1987, 1989), Scarsini and Shaked (1996), Joe (1997).

Theorem 4.1. *Let* $\mathbf{X}, \mathbf{Y} \in \mathcal{X}_+^2(\boldsymbol{\mu}, P)$. *If* $\mathbf{X} \leq_{\text{conc}} \mathbf{Y}$*, then* $\mathbf{X} \leq_{\text{ld}} \mathbf{Y}$.

Definition 4.2. Let Q', Q'' be two probability measures on $(\mathbb{R}^2, \text{Bor}(\mathbb{R}^2))$ with finite support. We say that Q' is obtained from Q'' via a concordance-increasing transfer (CIT) if, for some $\varepsilon > 0$, and some $s_1 < t_1$ and $s_2 > t_2$,

$$\begin{aligned} Q''(\mathbf{s}) &= Q'(\mathbf{s}) - \varepsilon, \\ Q''(\mathbf{t}) &= Q'(\mathbf{t}) - \varepsilon, \\ Q''(\mathbf{s} \vee \mathbf{t}) &= Q'(\mathbf{s} \vee \mathbf{t}) + \varepsilon, \\ Q''(\mathbf{s} \wedge \mathbf{t}) &= Q'(\mathbf{s} \wedge \mathbf{t}) + \varepsilon, \\ Q''(\mathbf{z}) &= Q'(\mathbf{z}) \quad \forall \mathbf{z} \notin \{s_1, t_1\} \times \{s_2, t_2\}. \end{aligned}$$

Lemma 4.3 (Theorem 1 of Tchen (1980)). *Let the support of* $\mathbf{X}$ *and* $\mathbf{Y}$ *be finite. Then* $\mathbf{X} \leq_{\text{conc}} \mathbf{Y}$ *if and only if the distribution of* $\mathbf{Y}$ *can be obtained from the distribution of* $\mathbf{X}$ *through a finite sequence of concordance increasing transfers.*

Proof of Theorem 4.1. First we prove the result for distributions with finite support. By using Lemma 4.3 we only have to prove that a CIT shrinks the zonotope. Consider the segments that generate the zonoid $Z(\mathbf{Y})$, and choose two of them $\overline{\mathbf{0}, \mathbf{s}}$ and $\overline{\mathbf{0}, \mathbf{t}}$ such that $s_1 < t_1$, $s_2 > t_2$. Replace the segments $\overline{\mathbf{0}, \epsilon\mathbf{s}}$ and $\overline{\mathbf{0}, \epsilon\mathbf{t}}$ with the segments $\overline{\mathbf{0}, \epsilon(\mathbf{s} \vee \mathbf{t})}$ and $\overline{\mathbf{0}, \epsilon(\mathbf{s} \wedge \mathbf{t})}$, which corresponds to a CIT on the law of $\mathbf{Y}$. The Minkowski sum of $\overline{\mathbf{0}, \epsilon(\mathbf{s} \vee \mathbf{t})}$ and $\overline{\mathbf{0}, \epsilon(\mathbf{s} \wedge \mathbf{t})}$ is contained in the Minkowski sum of $\overline{\mathbf{0}, \epsilon\mathbf{s}}$ and $\overline{\mathbf{0}, \epsilon\mathbf{t}}$, which proves that the replacement has shrunk the zonoid.

Since any distribution can be approximated in law by a sequence of distributions with finite support, we have that the result holds in general. □

5 Some examples

5.1 Marshall-Olkin distribution

The following "fatal shock" model was introduced by Marshall and Olkin (1967). Suppose that the components of a two-component system die after receiving a shock which is always fatal. A shock to the first (resp. second) component occur at a random time following an exponential distribution with parameter λ_1 (resp. λ_2). An exponential distribution with parameter λ_{12} governs the occurrence of a fatal shock that kills both components. All the above distributions are mutually independent. Thus, if (X_1, X_2) denotes the lifetimes of the two components, the survival function is given by

$$P(X_1 > x_1, X_2 > x_2) = \exp\{-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_{12} \max(x_1, x_2)\}. \tag{5.1}$$

Marshall and Olkin show, among other features, that the distribution (5.1) can be decomposed into two parts: an absolutely continous part spread over $\mathbb{R}^2_+$ and a singular part located on the line $x_2 = x_1\,(> 0)$, reflecting the occurrence of a fatal shock for both components. Also, it is easy to verify that

$$E[X_1] = \frac{1}{\lambda_1 + \lambda_{12}} \qquad \text{and} \qquad E[X_2] = \frac{1}{\lambda_2 + \lambda_{12}}. \tag{5.2}$$

Consider now the exchangeable case $\lambda_1 = \lambda_2 =: \lambda$. Fix $K := \lambda + \lambda_{12}$, and consider the whole class of exchangeable bivariate distributions of the above type with expected value equal to $(1/K, 1/K)$. Let $\mathbf{X}^\lambda = (X_1^\lambda, X_2^\lambda)$ be a random vector with the above distribution. Intuitively, for fixed K, as λ decreases (and λ_{12} increases) we should record a "higher chance" for the fatal shock to both components and, therefore, a greater dependence between X_1^λ and X_2^λ should take place. Indeed we have that

$$\begin{aligned}&P((X_1^\lambda \le x_1, X_2^\lambda \le x_2) \cup (X_1^\lambda > x_1, X_2^\lambda > x_2)) = \\ &\quad 1 - \exp\{-K\, x_1\} - \exp\{-K\, x_2\} + 2\exp\{-\lambda(x_1 + x_2) - (K - \lambda)\max(x_1, x_2)\}.\end{aligned}$$

is decreasing in λ. As a consequence of Theorem 4.1 we have

$$\lambda_A > \lambda_B \Longrightarrow \mathbf{X}^{\lambda_A} \le_{\text{conc}} \mathbf{X}^{\lambda_B} \Longrightarrow \mathbf{X}^{\lambda_A} \le_{\text{ld}} \mathbf{X}^{\lambda_B}.$$

If we examine a three-component fatal shock model, however, concordance cannot be used, and we must take a closer look at the zonoids themselves. Consider the following trivariate extension of the exchangeable fatal shock model

$$P(X_1^\lambda > x_1, X_2^\lambda > x_2, X_3^\lambda > x_3) = \exp\{-\lambda x_1 - \lambda x_2 - \lambda x_3 - \lambda_{123} \max(x_1, x_2, x_3)\}. \tag{5.3}$$

Here shocks kill either a single component or the three of them (shocks that leave only one component running are ruled out). It can be shown that the distribution of the absolutely continuous part is given by

$$f_{\mathrm{a}}(x_1, x_2, x_3) = \lambda^2(\lambda + \lambda_{123}) \exp\{-\lambda x_1 - \lambda x_2 - \lambda x_3 - \lambda_{123} \max\{x_1, x_2, x_3\}\}$$

The singular part occurs when the fatal shock kills all components. In case the triple shock hits first, the event is distributed on the line $\{(x_1, x_2, x_3) : x_1 = x_2 = x_3 = t\}$, with density

$$f_{\mathrm{s}}(t) = \lambda_{123} \exp\{-(3\lambda + \lambda_{123})t\}.$$

Also, it may happen that a single component, say the first, is killed and then the triple shock occurs, bringing down the remaining two components. In such circumstance, the distribution lies on the region $\{(x_1, x_2, x_3) : x_1 < x_2 = x_3 = t\}$, with density

$$f_{\mathrm{d}_1}(x_1, t) = \lambda\, \lambda_{123} \exp\{-(2\lambda + \lambda_{123})t - \lambda\, x_1\}.$$

Similarly, we can derive the density $f_{\mathrm{d}_2}(x_2, t)$ (resp. $f_{\mathrm{d}_3}(x_3, t)$) that models the situation where the second (resp. third) component is the first to die.

If we fix $\lambda + \lambda_{123} = K$, we have, again $E[X_i] = K^{-1}$ for $1 = 1, 2, 3$. An important tool to study the zonoids of $\mathbf{X}^\lambda = (X_1^\lambda, X_2^\lambda, X_3^\lambda)$, as λ ranges between 0 and K, is given by the support function $h(Z(\mathbf{X}^\lambda), \cdot)$, defined in Section 2.

It is known (see Schneider (1993)) that $Z(\mathbf{X}) \subset Z(\mathbf{X}')$ occurs if and only if $h(Z(\mathbf{X}), \mathbf{p}) \leq h(Z(\mathbf{X}'), \mathbf{p})$ for all $\mathbf{p} \in \mathcal{S}^2$.

When $p_1, p_2, p_3 > 0$, the support function takes the value

$$h(\mathbf{X}^\lambda, \mathbf{p}) = \frac{p_1 + p_2 + p_3}{K}$$

and this value is not affected by changes of λ. A similar behavior holds when $p_1, p_2, p_3 < 0$, since then $h(\mathbf{X}^\lambda, \mathbf{p}) = 0$.

More interesting are the cases where the components in $\mathbf{p}$ have different signs. Due to the symmetry in the distribution we can focus on the situation where $p_1, p_2 > 0$, $p_3 < 0$ and $\langle \mathbf{p}, \mathbf{p} \rangle = 1$. All the other cases can be derived from this particular setting. Four different subcases can be singled out. The first one is $p_1 > -p_3$ and $p_2 > -p_3$. The zonoid is given by

$$\begin{aligned} h(\mathbf{X}^\lambda, \mathbf{p}) = \big(&\lambda^2 p_2 p_3^2 + \lambda^2 p_1 p_3^2 - \lambda K p_2^2 p_3 - 2\lambda K p_1 p_2 p_3 - \lambda K p_2 p_3^2 - \lambda K p_1^2 p_3 - \\ &\lambda K p_1 p_3^2 + K^2 p_1 p_2^2 + K^2 p_1^2 p_2 + K^2 p_1 p_2 p_3\big) / (K(\lambda p_3 - K p_1)(\lambda p_3 - K p_2)). \end{aligned}$$

By taking derivative with respect to λ we obtain

$$\frac{p_3^3\lambda(\lambda p_1p_3 + \lambda p_2p_3 - 2Kp_1p_2)}{(\lambda p_3 - Kp_1)^2(\lambda p_3 - Kp_2)^2} > 0$$

Therefore

$$\lambda_A > \lambda_B \Rightarrow h(Z(\mathbf{X}^{\lambda_A}), \mathbf{p}) \geq h(Z(\mathbf{X}^{\lambda_B}), \mathbf{p}) \tag{5.4}$$

Two other cases can be computed analytically, even if they are quite cumbersome, but the last one requires numerical evaluation, which was performed with the *Maple V* programming language. For all these cases (5.4) holds. It is interesting to see that the linear order that we defined in terms of zonoids agrees with the intuitive idea that if the probability of a common shock killing all the components increases, then the linear dependence between the lifetimes increases.

5.2 A copula model

Consider a triplet of random variables $\mathbf{X}_\alpha = (X_1^\alpha, X_2^\alpha, X_3^\alpha)$ whose distribution function is a trivariate copula, i.e. a distribution on $[0,1]^3$ with uniform marginals. In particular let

$$H_\alpha(x_1, x_2, x_3) = \alpha H^+(x_1, x_2, x_3) + (1-\alpha)H^\perp(x_1, x_2, x_3)$$

where

$$H^+(x_1, x_2, x_3) = \min(x_1, x_2, x_3) I_{[0,1]^3}(x_1, x_2, x_3),$$

is the upper Fréchet bound, and

$$H^\perp(x_1, x_2, x_3) = x_1x_2x_3 I_{[0,1]^3}(x_1, x_2, x_3),$$

is the independent copula. Therefore H_α is a mixture between the independent distribution and the upper Fréchet bound in the class of trivariate copulae.

If $\mathbf{X}$, $\mathbf{X}_1$ and $\mathbf{X}_2$ have distribution functions F, F_1 and F_2, respectively, and $F = \alpha F_1 + (1-\alpha)F_2$, then

$$Z(\mathbf{X}) = \alpha Z(\mathbf{X}_1) \oplus (1-\alpha)Z(\mathbf{X}_2)$$

(see e.g. Koshevoy and Mosler (1998, Theorem 3.2)). Also, it is easy to show that the zonoid Z^+ associated to H^+ is the segment $\overline{\mathbf{0},\mathbf{1}}$. Call $Z^\perp$ the zonoid associated with $H^{\perp}$. Then, for $\alpha > \beta$,

$$Z(\mathbf{X}_\alpha) = \alpha Z^+ \oplus (1-\alpha) Z^{\perp} = \beta Z^+ \oplus (\alpha - \beta) Z^+ \oplus (1-\alpha) Z^{\perp} \subset \\ \beta Z^+ \oplus (\alpha - \beta) Z^{\perp} \oplus (1-\alpha) Z^{\perp} = \beta Z^+ \oplus (1-\beta) Z^{\perp} = Z(\mathbf{X}_\beta)$$

and $\mathbf{X}_\alpha \geq_{\mathrm{ld}} \mathbf{X}_\beta$. Thus, $\mathbf{X}_\alpha$ is increasing in linear dependence in the parameter α.

6 Measures of linear dependence

Since we have defined in Section 3 an order of linear dependence based on the inclusion of zonoids, it is quite conceivable to define measures of linear dependence based on the volume of these zonoids. In particular if we define for $X \in \mathcal{X}^d_+(\boldsymbol{\mu}, P)$,

$$D(\mathbf{X}) = 1 - \frac{\mathrm{vol}[Z_P(\mathbf{X})]}{\prod_{i=1}^d \mu_i} \tag{6.1}$$

we have that D is increasing in the order $\leq_{\mathrm{ld}}$, with $D(\mathbf{X}^+) = 1$, and $D(\mathbf{X}^-) = 0$.

A linear dependence measure like D presents the shortcoming of achieving its maximum value 1 not only when

$$\frac{X_1}{\mu_1} = \frac{X_2}{\mu_2} = \cdots = \frac{X_d}{\mu_d}.$$

Actually, whenever for some $i, j = 1, \ldots, d$, $i \neq j$, we have $X_i/\mu_i = X_j/\mu_j$, then $D(\mathbf{X}) = 1$.

This shortcoming could be overcome by adapting a method proposed by Koshevoy and Mosler (1997) for measures of multivariate inquality. We define a new class of indices

$$D_\varepsilon(\mathbf{X}) = 1 - \frac{\mathrm{vol}[Z_P(\mathbf{X}) \oplus C^d_\varepsilon] - \varepsilon^{d-1}\left(\sum_{i=1}^d \mu_i + \varepsilon\right)}{\prod_{i=1}^d (\mu_i + \varepsilon) - \varepsilon^{d-1}\left(\sum_{i=1}^d \mu_i + \varepsilon\right)},$$

where $C^d_\varepsilon = [0, \varepsilon]^d$ is the d-dimensional cube of side-length ε, for some $\epsilon > 0$. We still have

$$D_\varepsilon(\mathbf{X}^+) = 1, \qquad D_\varepsilon(\mathbf{X}^-) = 0, \qquad \forall \varepsilon > 0,$$

and $D_\varepsilon(\cdot)$ is increasing in $\leq_{\mathrm{ld}}$. But now $D_\varepsilon(\mathbf{X})$ is sensitive to the dimension of the subspace on which $Z_P(\mathbf{X})$ lies. The larger ε, the larger the sensitivity.

For $d = 2$ the index D works without the above mentioned problem, and can be related to the mean difference of $\widetilde{X}_1$ as the following theorem shows.

Theorem 6.1. *Let* $\mathbf{X} \in \mathcal{X}_+^2(\boldsymbol{\mu}, P)$

$$D(\mathbf{X}) = \frac{2\mu_1\mu_2 - (\mu_1+\mu_2)^2 E_{Q^{\mathbf{X}}}[|\widetilde{X}_1 - \widetilde{W}_1|]}{2\mu_1\mu_2},$$

where $\widetilde{W}_1$ *is an independent copy of* $\widetilde{X}_1$ *under* $Q^{\mathbf{X}}$*.*

The proof of Theorem 6.1 requires the following two lemmata. The first is a classical well known result that relates the volume of the lift zonoid and the mean difference, and goes back to Gini (1914) (for a recent reference see e.g. Koshevoy and Mosler (1997)).

Lemma 6.2.

$$E_{Q^{\mathbf{X}}}[|\widetilde{X}_1 - \widetilde{W}_1|] = 2\,\mathrm{vol}[\ell_{Q^{\mathbf{X}}}(\widetilde{X}_1)]$$

Given a random vector $\mathbf{X} = (X_1, \ldots, X_d)$ we define $\mathbf{X}_{-i} = (X_1, \ldots, X_{i-1}, X_{i+1}, X_d)$.

Lemma 6.3. *For* $i = 1, \ldots, d$

$$\mathrm{vol}[Z_P(\mathbf{X})] = \left(\sum_{j=1}^{d} \mu_j\right)^d \mathrm{vol}[\ell_{Q^{\mathbf{X}}}(\widetilde{\mathbf{X}}_{-i})].$$

Proof. By Theorem 2.4 $Z_P(\mathbf{X}) = \left(\sum_{i=1}^{d} \mu_i\right) Z_{Q^{\mathbf{X}}}(\widetilde{\mathbf{X}})$. Therefore all we have to prove is that

$$\mathrm{vol}[Z_{Q^{\mathbf{X}}}(\widetilde{\mathbf{X}})] = \mathrm{vol}[\ell_{Q^{\mathbf{X}}}(\widetilde{\mathbf{X}}_{-i})], \qquad i = 1, \ldots, d.$$

Given a vector $\mathbf{t} \in \Sigma_{d-1}$, consider the sets

$$T_i = \{\tau \in \mathbb{R} : (t_1, \ldots, t_{i-1}, \tau, t_{i+1}, \ldots, t_d) \in Z_{Q^{\mathbf{X}}}(\widetilde{\mathbf{X}})\},$$

and

$$S_i = \{s \in \mathbb{R} : (s, t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_d) \in \ell_{Q^{\mathbf{X}}}(\widetilde{\mathbf{X}}_{-i})\}.$$

If we define, for $i = 1, \ldots, d$, $Q^{\mathbf{X}}_{\widetilde{\mathbf{X}}_i}$ as

$$Q^{\mathbf{X}}_{\widetilde{\mathbf{X}}_i}(A) = \int_A \widetilde{\mathbf{X}}_i(\omega)\; Q^{\mathbf{X}}(d\omega),$$

we have $Q^{\mathbf{X}}(A) = \sum_{i=1}^{d} Q^{\mathbf{X}}_{\widetilde{\mathbf{X}}_i}(A)$. Hence

$$S_i = T_i + \sum_{\substack{j=1 \\ j \neq i}}^{d} t_j.$$

Therefore, for $i = 1, \dots, d$, $\mathrm{Leb}(T_i) = \mathrm{Leb}(S_i)$, which implies

$$\mathrm{vol}[Z_{Q^{\mathbf{X}}}(\widetilde{\mathbf{X}})] = \mathrm{vol}[\ell_{Q^{\mathbf{X}}}(\widetilde{\mathbf{X}}_{-i})], \qquad i = 1, \dots, d.$$

□

Proof of Theorem 6.1. The result is an immediate consequence of the combination of Lemma 6.2, Lemma 6.3 (with $d = 2$ and $i = 1$), and (6.1). □

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