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Positive value of information in games*

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Abstract

We exhibit a general class of interactive decision situations in which all the agents benefit from more information. This class includes as a special case the classical comparison of statistical experiments *à la* Blackwell.

More specifically, we consider pairs consisting of a game with incomplete information G and an information structure \mathcal{S} such that the extended game $\Gamma(G, \mathcal{S})$ has a unique Pareto payoff profile u . We prove that u is a Nash payoff profile of $\Gamma(G, \mathcal{S})$, and that for any information structure \mathcal{T} that is coarser than \mathcal{S} , all Nash payoff profiles of $\Gamma(G, \mathcal{T})$ are dominated by u . We then prove that our condition is also necessary in the following sense: Given any convex compact polyhedron of payoff profiles, whose Pareto frontier is not a singleton, there exists an extended game $\Gamma(G, \mathcal{S})$ with that polyhedron as the convex hull of feasible payoffs, an information structure \mathcal{T} coarser than \mathcal{S} and a player i who strictly prefers a Nash equilibrium in $\Gamma(G, \mathcal{T})$ to any Nash equilibrium in $\Gamma(G, \mathcal{S})$.

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1 Introduction

Blackwell's theory of *comparison of statistical experiments* shows that for single-agent decision problems, more information to the agent is always better (see Blackwell (1951, 1953)). The formalization of similar ideas actually goes back to Ramsey, whose note on the topic has been recently published (Ramsey (1990)). However, monotonicity of payoffs with respect to information fails to hold in many cases. For instance Kadane, Schervish, and Seidenfeld (1996) have shown that a Bayesian decision maker may rationally pay not to see the outcome of a certain cost-free experiment, when only finite additivity of the probability measures is assumed. Wakker (1988), Schlee (1990, 1991), Safra and Sulganik (1995), Chassagnon and Vergnaud (1999) among others have dealt with similar phenomena for decision makers whose choice criterion is not the maximization of expected utility. Sulganik and Zilcha (1997) have shown that information is not always beneficial if the feasible set of actions depends on the signal and the information system. Building on the work of Kreps and Porteus (1978), Grant, Kajii, and Polak (1998a, 1998b) consider intrinsic preference for information, and provide conditions for a dynamically consistent agent always to prefer more informative signals. Lehmann (1988) weakened Blackwell's criterion for the comparison of experiments. His idea has been used by Athey and Levin (1998) and Persico (2000) to compare attitudes towards information in some classes of models.

In the context of more than one decision maker, i.e. in game theory, the negative value of information and its economic implications were examined by Hirschleifer (1971). Kamien, Tauman, and Zamir (1990a, 1990b) considered games in which an outside agent can send signals to the players. The effect of information was captured by the equilibria of the games induced by these messages. Neyman (1991) emphasized the fact that more information to a player cannot have a negative effect as long as other players are not aware of it. Gossner (2000) compared information structures according to the correlated equilibrium distributions they induce in games with incomplete information.

Recall the following simple instance of information rejection (see e.g. Kamien, Tauman, and Zamir (1990)). A card is drawn at random from a deck, and it can be either red or black, with equal probabilities. Player I declares a color and player II listens to what player I says, and then declares a color. If both players declare the same color, they get two dollars each. Otherwise the player, whose declared color matches the color of the drawn card, gets six dollars, and the other player gets zero. It is clear that, if the first player gets to see the card before making her declaration, then her dominant strategy is to declare the color of the drawn card. If it is common knowledge that the first player saw the card before making her declaration, then the best reply of the second player is to declare what the first player declared. The equilibrium leads to a payoff profile of two dollars each. If the first player does not get to see the card before making her declaration, and this is common knowledge, then any strategy is equivalent for her, and the dominant strategy for the second player is

to declare the other color. The equilibrium expected payoff is three dollars each. In this game it is better for both players that no information be made available to the first player.

The issue of negative value of information bears some analogy with the following so-called Braess paradox (see e.g. Cohen and Kelly (1990), and Bean, Kelly, and Taylor (1997)): In non-cooperative networks it is possible that the addition of resources to the network is accompanied by a degradation of the performance. Korilis, Lazar, and Orda (1999) explained how the paradox is due to the non-cooperative structure of the network and to the fact that in general Nash equilibria are not Pareto efficient. Furthermore they found conditions under which the paradox cannot happen.

Although it seems that the typical situation in game theory is that the ‘expected payoff profile’ is not necessarily improved when agents get more informed, we prove that under some conditions (which include as a special case the one-agent maximization problem) the value of information is positive.

We consider a game with incomplete information and an information structure such that the extended game possesses a unique Pareto payoff. We remark that this Pareto payoff is necessarily a Nash payoff of this game, and we show that every Nash payoff induced by any coarser information structure is Pareto dominated by this payoff profile. The class of pairs (game and information structure) that we consider is the one with the common interest property of Aumann and Sorin (1989).

Vice versa we show that for any closed convex polyhedron Δ whose Pareto frontier is not a singleton there exists an extended game such that the convex hull of the set of its feasible payoffs is the above set Δ and the value of information is not positive for at least one player. This provides a full characterization of a property that we call *positive value of information*, and shows that we cannot get rid of uniqueness of the Pareto payoff if we want to be sure that information has a positive value.

Our condition on the pairs (game, information structure) is thus on the set of feasible payoffs of the extended (normal form) game. Note that some modifications of a game –like permutations of payoffs associated to outcomes– would affect its set of equilibria, but not its set of feasible payoffs. Our condition would then be unmodified under such a transformation. Indeed, when a game admits a unique Pareto payoff, it becomes all player’s incentives to reach this point, and these incentives do not depend on the exact strategic form of the underlying game. Remark also that geometric properties of the set of feasible payoff can be informative about the strategic properties of underlying game. For instance, with two players, if this set is a line with negative slope, the corresponding game must essentially be zero-sum.

2 Some examples

In this section we provide some examples exhibiting various effects of information. Many more can be found in Bassan, Scarsini, and Zamir (1997), from which these examples are drawn. Since we consider matrix games, Nash equilibrium always exists

(in mixed strategies). Typically, there are several equilibria and there is no “obvious outcome” of the game. The games that we consider are all solvable by iterative deletion of dominated strategies, thus we avoid all the difficulties that may rise due to the existence of multiple Nash equilibria, and we use the expression “*the outcome of the game*”.

In all the following examples, nature chooses one of the two matrices G_A, G_B with probability $1/2$, the interpretation being that the state of nature is either A or B with equal probabilities. If the state is A (respectively: B), the payoff matrix is G_A (respectively: G_B). We shall refer to G_A and G_B as *state-games*. The state-games are given in normal (strategic) form (the choices are made simultaneously). An entry (a, b) represents a payoff of a units to the row player and b units to the column player.

Each example consists of four games corresponding to four different information structures.

Example 1.

$$G_A = \begin{array}{c} T \\ B \end{array} \begin{array}{cc} L & R \\ \hline 0, 0 & 6, -3 \\ -3, 6 & 5, 5 \end{array} \quad G_B = \begin{array}{c} T \\ B \end{array} \begin{array}{cc} L & R \\ \hline -20, -20 & -7, -16 \\ -16, -7 & -5, -5 \end{array}$$

We refer to the rows as T (top) and B (bottom) and to the columns as L (left) and R (right). The two matrixes are common knowledge, and so is the fact that nature chooses one of them with probability $1/2$. First of all notice that in G_A the top row, T , strongly dominates the bottom row, B , and the left column, L , strongly dominates the right column, R . In G_B the situation is reversed: B strongly dominates T and R strongly dominates L . Therefore $(0, 0)$ is the unique Nash equilibrium payoff in G_A and $(-5, -5)$ is the unique Nash equilibrium payoff in G_B . From this it follows that, if, before the players make their move, the state-game is revealed to both of them, they expect a payoff equal to

$$\frac{1}{2}(0, 0) + \frac{1}{2}(-5, -5) = (-2.5, -2.5).$$

If it is common knowledge that both players are uninformed about the state-game that is being played, they act as if they were playing the game

$$\frac{1}{2}G_A + \frac{1}{2}G_B = \begin{array}{c} T \\ B \end{array} \begin{array}{cc} L & R \\ \hline -10, -10 & -0.5, -9.5 \\ -9.5, -0.5 & 0, 0 \end{array}$$

In this game B strongly dominates T and R strongly dominates L . Therefore the unique Nash equilibrium payoff is $(0, 0)$.

If it is common knowledge that the state-game is revealed only to the row player, then the column player will expect the row player to choose T in G_A and B in G_B .

Therefore the payoffs are (depending whether the column player chooses L or R).

$$\begin{array}{cc} L & R \\ \boxed{0, 0} & \boxed{6, -3} \end{array} \quad \text{with probability } 1/2,$$

and

$$\begin{array}{cc} L & R \\ \boxed{-16, -7} & \boxed{-5, -5} \end{array} \quad \text{with probability } 1/2,$$

Hence she will have to choose left or right in the following row of expected payoffs

$$\boxed{-8, -3.5} \quad \boxed{0.5, -4}$$

The outcome of the game is now $(-8, -3.5)$.

By using the symmetry of the games, we can see that, if only the column player is informed, and this is common knowledge, then the outcome is $(-3.5, -8)$.

If we summarize these results in what we shall refer to as the I-U (Informed-Uninformed) matrix,

$$\begin{array}{cc} & \text{2-Inf} & \text{2-Uninf} \\ \text{1-Inf} & \boxed{-2.5, -2.5} & \boxed{-8, -3.5} \\ \text{1-Uninf} & \boxed{-3.5, -8} & \boxed{0, 0} \end{array}$$

we immediately see that the situation in which both players are uninformed is strongly preferred by both of them to all other three situations.

Example 2.

$$G_A = \begin{array}{cc} & L & R \\ T & \boxed{2, 2} & \boxed{-1, -6} \\ B & \boxed{-6, -1} & \boxed{-2, -2} \end{array} \quad G_B = \begin{array}{cc} & L & R \\ T & \boxed{-20, -20} & \boxed{-5, -5} \\ B & \boxed{-5, -5} & \boxed{2, 2} \end{array}$$

This example can be solved along the lines of the previous one, and it is easily verified that the I-U matrix is

$$\begin{array}{cc} & \text{2-Inf} & \text{2-Uninf} \\ \text{1-Inf} & \boxed{2, 2} & \boxed{-1.5, -1.5} \\ \text{1-Uninf} & \boxed{-1.5, -1.5} & \boxed{0, 0} \end{array}$$

Information is good for both, if both have it, bad for both, if only one has it. Put differently, the information of the two players is *complementary* to each other. The reason for this complementarity, as can be seen from the matrices, is that in order to take advantage of the knowledge about the state of nature, they have to coordinate, and to do that they both have to know the state.

3 A characterization result

We consider a set of agents I (with an abuse of notation we define I to be $\{1, 2, \dots, I\}$), a probability space (Ω, \mathcal{F}, P) , a measurable mapping κ from Ω to some measurable space (K, \mathcal{K}) (the parameter set, or the set of *states of nature*, which, as in Mertens, Sorin, and Zamir (1994) is fixed), and a family $(\mathcal{S}_i)_{i \in I}$ of sub σ -fields of \mathcal{F} . We call $\mathcal{S} = (\Omega, \mathcal{F}, (\mathcal{S}_i)_{i \in I}, \kappa, P)$ an *information structure*. The σ -field \mathcal{S}_i is interpreted as the information available to agent i . To be precise the definition of \mathcal{S} should depend also on (K, \mathcal{K}) . For the sake of simplicity we omit the indication of this dependence, also because, as we said, it is assumed to be the same throughout the paper.

Given $\mathcal{S} = (\Omega, \mathcal{F}, (\mathcal{S}_i)_{i \in I}, \kappa, P)$ and $\mathcal{T} = (\Omega, \mathcal{F}, (\mathcal{T}_i)_{i \in I}, \kappa, P)$, we say that \mathcal{S} is more informative for all players than \mathcal{T} , and we write $\mathcal{S} \supseteq \mathcal{T}$, when $\mathcal{S}_i \supseteq \mathcal{T}_i$ for all $i \in I$. The set inclusion $\mathcal{S}_i \supseteq \mathcal{T}_i$ means that whatever player i knows under \mathcal{T}_i , she knows also under \mathcal{S}_i .

A *game of incomplete information* G is given by a family of measurable sets $(A_i)_{i \in I}$ and by a measurable and bounded payoff function $g : \times_{i \in I} A_i \times K \rightarrow \mathbb{R}^I$.

An information structure \mathcal{S} together with a game of incomplete information (with the same state space K), defines an extended (Bayesian) game $\Gamma(G, \mathcal{S})$. In this game, the set of strategies $\Sigma_i^{\mathcal{S}}$ for player i is the set of \mathcal{S}_i -measurable functions $f_i : \Omega \rightarrow A_i$, and the payoff function is defined by the relation $g_{\mathcal{S}}(f) = E_P g((f(\cdot)), \kappa(\cdot))$, where $f = (f_i)_{i \in I}$. (The boundedness of g is to ensure that the expectation is well defined.) We denote $\Sigma^{\mathcal{S}} = \times_{i \in I} \Sigma_i^{\mathcal{S}}$. We let $F(G, \mathcal{S})$ denote the set of feasible payoffs of $\Gamma(G, \mathcal{S})$. Given two points x, y in \mathbb{R}^I , we write $y \geq x$ when for every i , $y_i \geq x_i$. The following definitions are needed in the sequel.

Definition 3. We say that x (Pareto) dominates y if $x \geq y$. Given a subset B of \mathbb{R}^I , we denote by $\text{Pa}(B)$ its Pareto frontier, namely,

$$x \in \text{Pa}(B) \subseteq B \text{ iff } \nexists y \in B \text{ such that } y \neq x \text{ and } y \geq x.$$

Definition 4. For a closed set B define $\text{co}(B)$ its convex hull, and $\text{Ex}(B)$ the set of the extreme points of $\text{co}(B)$.

Definition 5. A game $\Gamma(G, \mathcal{S})$ has the *positive-value-of-information property* (PVIP) if, whenever $\mathcal{S} \supseteq \mathcal{T}$, every Nash equilibrium payoff of $\Gamma(G, \mathcal{T})$ is dominated by a Nash equilibrium payoff of $\Gamma(G, \mathcal{S})$.

Our result of monotonicity of payoff with respect to information is the following:

Theorem 6. *Let Δ be a closed convex subset of \mathbb{R}^I with a finite number of extreme points. Then the following are equivalent:*

- (i) *All games $\Gamma(G, \mathcal{S})$ such that $F(G, \mathcal{S})$ is closed and $\text{co}(F(G, \mathcal{S})) = \Delta$ have the PVIP,*

(ii) $\text{Pa}(\Delta)$ is a singleton.

Proof. First we prove that (ii) implies (i). Let $\text{Pa}(\Delta) = \{v\}$, and let (G, \mathcal{S}) be such that $F(G, \mathcal{S})$ is closed and $\text{co}(F(G, \mathcal{S})) = \Delta$.

Since $F(G, \mathcal{S})$ is closed, it contains all its extreme points, hence $v \in F(G, \mathcal{S})$: $v = g_{\mathcal{S}}(f_0)$ for some $f_0 \in \Sigma^{\mathcal{S}}$. Notice that f_0 is a Nash equilibrium of $\Gamma(G, \mathcal{S})$: indeed, all feasible payoffs of $\Gamma(G, \mathcal{S})$ are dominated by v .

Consider now \mathcal{T} such that $\mathcal{S} \supseteq \mathcal{T}$. Remark that

- $\Sigma_i^{\mathcal{T}} \subseteq \Sigma_i^{\mathcal{S}}$, for all $i \in I$, and
- for any $f \in \Sigma^{\mathcal{T}}$, we have $g_{\mathcal{T}}(f) = g_{\mathcal{S}}(f)$.

Hence all feasible payoffs (and in particular all Nash payoffs) of $\Gamma(G, \mathcal{T})$ are dominated by v .

Now we prove that not (ii) implies not (i), namely, that if $\text{Pa}(\Delta)$ is not a singleton, then there exists a game $\Gamma(G, \mathcal{S})$ such that $F(G, \mathcal{S})$ is closed, $\text{co}(F(G, \mathcal{S})) = \Delta$, and $\Gamma(G, \mathcal{S})$ does not have the PVIP.

Take $K = \text{Ex}(\Delta)$, and define inductively for $i \in I$,

$$\begin{aligned} K^0 &= K, \\ K^i &= \{k \in K^{i-1} : \forall h \in K^{i-1}, k_i \geq h_i\}. \end{aligned}$$

Points in K^i are thus the points in K^{i-1} preferred by player i . By definition $K^i \subseteq K^{i-1}$, and $K^I = \bigcap_i K^i$. Note also that two points in K^i have the same i -coordinate, and hence also same j -coordinates for $j \leq i$. In particular, K^I is a singleton $\{\alpha\}$. Clearly $\alpha \in \text{Pa}(\Delta)$. In fact, if not, then there is $\gamma \in K$ such that $\gamma \geq \alpha$ and $\gamma \neq \alpha$. Let $i_0 \in I$ be such that $\gamma_i = \alpha_i$ for all $i < i_0$ and $\gamma_{i_0} > \alpha_{i_0}$. This would imply $\alpha \notin K^{i_0}$, a contradiction.

To continue the proof we need the following

Lemma 7. *There exists a game G_0 of complete information with finite strategy sets, with Δ as the convex hull of feasible payoffs, and with α as its unique Nash payoff.*

Proof. First, assume that for every i , $K^i \neq K^{i-1}$, and let $\gamma^i \in K^{i-1} \setminus K^i$.

Let G_0 be the game with complete information where player 1 chooses $k \in K$ and all the other players choose either c (continue) or s (stop). Given a strategy profile $(k; a_2, \dots, a_I)$; $a_j \in \{c, s\}$, the payoff is defined as follows:

(a) The payoff is k in each of the following cases:

- $a_j = s$; $\forall j \in I \setminus \{1\}$,
- $k = \alpha$,
- $k \in K \setminus K^1$.

(b) In all other cases, let m be such that $k \in K^{m-1} \setminus K^m$ (note that $m \geq 2$); follow the following procedure starting at stage m° .

2 $^\circ$ If player 2 chooses s , the payoff is γ^2 , if not go to 3 $^\circ$.

3 $^\circ$ If player 3 chooses s , the payoff is γ^3 , if not go to 4 $^\circ$.

...

i° If player i chooses s , the payoff is γ^i , if not go to $(i+1)^\circ$.

...

I° If player I chooses s , the payoff is γ^I , if not it is α .

To see that the convex hull of the set of feasible payoffs of G_0 is Δ , observe that the payoff is always a point in K . Furthermore any point $k \in K$ is a feasible payoff (obtained for example when 1 chooses k and all other players choose s).

Now we show that α is the only Nash payoff of G_0 . In particular we prove the following:

(A) For $2 \leq m \leq I$ and $k \in K^{m-1} \setminus K^m$ any strategy profile of the form (a_1, \dots, a_I) , with $a_1 = k$, $a_2, \dots, a_{m-1} \in \{c, s\}$, $a_m = a_{m+1} = \dots = a_I = c$, is a Nash equilibrium, whose payoff is α .

(B) The profile (a_1, \dots, a_I) with $a_1 = \alpha$ and $a_2, \dots, a_I \in \{c, s\}$ is a Nash equilibrium whose payoff is α .

(C) Any other strategy profile is not a Nash equilibrium.

First we prove (A). Clearly, such a strategy profile yields a payoff α . To see that it is an equilibrium we observe the following:

- Any player i such that $1 < i < m$ does not affect the payoff and hence has no profitable deviation,
- A player $i \geq m$ plays c . A deviation to s would yield an outcome γ^i , which for her is worse than α , hence she has no profitable deviation.
- Player 1 receives the same payoff α_1 for any choice $k \in K^1$ and any deviation to $k \in K \setminus K^1$ yields a lower payoff by definition of K^1 .

(B) is evident.

To prove (C), observe first that in any equilibrium player 1 has to choose $k \in K^1$ since any other k is strictly dominated. Once player 1 has chosen $k \in K^{m-1} \setminus K^m$, for $m \geq 2$, players $2, \dots, m-1$ are irrelevant and any player $i \geq m$ who chooses s receives a payoff γ^i which is worse for her than any payoff she might get by playing c .

To finish the proof, in case there exist some players j such that $K^j = K^{j-1}$, we simply modify the game G_0 above in such a way that the final payoff does not depend on the actions chosen by those players. \square

To continue the proof of “not (ii) implies not (i)”, let $\beta \in \text{Ex}(\Delta)$ be a point not dominated by α (such β exists since (ii) is not satisfied). Let i_0 be a player such that $\beta \in K^{i_0-1} \setminus K^{i_0}$.

We now define a game G_1 with incomplete information as follows:

- There are 4 states of the world, $R1, R2, B1, B2$, chosen with probabilities $p/2, (1-p)/2, (1-p)/2, p/2$ respectively, with $0 < p < 1/2$.
- In $R1, R2$ the state of nature is R while in $B1, B2$ the state of nature is B .
- Following the chance move choosing the state of the world (and the private information given to the players), player i_0 has to announce, publicly, the state of nature (R or B). If i_0 is right, G_0 is played. If i_0 is wrong, the outcome is β .

In the information structure \mathcal{S} , player i_0 is informed of the digit 1, 2 appearing in the state of the world, (but not of letter R or B). In the (coarser) information structure \mathcal{T} , i_0 receives no information about the state of the world. All other players receive no information, both in \mathcal{S} and in \mathcal{T} .

We proceed to prove that $\Gamma(G_1, \mathcal{S})$ does not have the PVIP: In G extended by \mathcal{S} or \mathcal{T} , player i_0 has no strategy that ensures her to be wrong with probability 1. Hence the probability that the subgame G_0 is played is positive for every strategy of i_0 . In particular, at all Nash equilibria of G_1 extended by \mathcal{S} or \mathcal{T} the outcome in the subgame G_0 is α (the only Nash equilibrium of G_0). This is reached with positive probability.

Since $\beta_{i_0} < \alpha_{i_0}$, player i_0 prefers to be right than wrong. So in $\Gamma(G_1, \mathcal{S})$ she will be right with probability $(1-p)$ while in $\Gamma(G_1, \mathcal{T})$ she will be right with probability $1/2$. It follows that

- in the game extended by $\Gamma(G_1, \mathcal{S})$ the only Nash payoff is $(1-p)\alpha + p\beta$,
- in the game extended by $\Gamma(G_1, \mathcal{T})$ the only Nash payoff is $\alpha/2 + \beta/2$,
- since α does not dominate β , the payoff $(1-p)\alpha + p\beta$ does not dominate $\alpha/2 + \beta/2$ and hence the game $\Gamma(G_1, \mathcal{S})$ does not have the PVIP.

The last stage of the proof is now to modify the game G_1 to G_2 so that $\Gamma(G_2, \mathcal{S})$ also does not have the PVIP and $\text{co}(G_2, \mathcal{S}) = \Delta$.

With any strategy she may use, player i_0 in (G_1, \mathcal{S}) will be right with probability $(1-p)$, $\frac{1}{2}$, or p . Hence

$$F(G_1, \mathcal{S}) = \{p\beta + (1-p)K\} \cup \{\frac{1}{2}\beta + \frac{1}{2}K\} \cup \{(1-p)\beta + pK\}$$

Since $\beta \in K$, this can be written as

$$F(G_1, \mathcal{S}) = \{\beta, p\beta + (1-p)K\} \cup \{\beta, \frac{1}{2}\beta + \frac{1}{2}K\} \cup \{\beta, (1-p)\beta + pK\},$$

and so

$$\text{co}(F(G_1, \mathcal{S})) = \text{co} \left\{ \text{co}\{\beta, p\beta + (1-p)K\} \cup \text{co}\{\beta, \frac{1}{2}\beta + \frac{1}{2}K\} \cup \text{co}\{\beta, (1-p)\beta + pK\} \right\}$$

Since $p < (1-p)$, $1/2 < (1-p)$,

$$\text{co}\{\beta, (1-p)\beta + pK\} \subset \text{co}\{\beta, p\beta + (1-p)K\} = \text{co}\{p\beta + (1-p)K\},$$

and

$$\text{co}\{\beta, \frac{1}{2}\beta + \frac{1}{2}K\} \subset \text{co}\{\beta, p\beta + (1-p)K\} = \text{co}\{p\beta + (1-p)K\},$$

we conclude that

$$\text{co}(F(G_1, \mathcal{S})) = \text{co}\{p\beta + (1-p)K\} = p\beta + (1-p)\text{co}(K) = p\beta + (1-p)\Delta.$$

Notice now that the PVIP is preserved under positive linear transformation of the payoffs. Thus in the game G_2 which is obtained from G_1 by the positive linear transformation of the payoffs:

$$x \rightarrow \frac{x - p\beta}{1-p}$$

we have

- $F(G_2, \mathcal{S})$ is closed (being finite) and $\text{co}(F(G_2, \mathcal{S})) = \Delta$,
- $\Gamma(G_2, \mathcal{S})$ does not have the PVIP,

completing the proof of Theorem 6. □

The rationale of the part “(ii) implies (i)” of the theorem is that any payoff that can be obtained under an information structure, can also be obtained under an information structure which is more informative for all agents (it’s enough to ignore the additional information). In this respect the multi-agent situation does not differ from the one-agent case. What is different is that the property of being an equilibrium in general is not preserved when going to more informative information structures. This is the origin of the many information paradoxes found in the literature. The assumption of uniqueness of Pareto payoff under the richer information structure is the key to avoid the paradoxes: More information accompanied with a unique Pareto payoff *does* yield a ‘better’ outcome. It is clear that our assumptions include in particular the case of one single player: In this case the Pareto payoff is always unique. This is coherent with Blackwell’s idea that more information is always better for a single decision maker.

The “not (ii) implies not (i)” part says that the uniqueness of the Pareto payoff, albeit a strong property, cannot be disposed of, if we want to insure PVIP. Whenever the Pareto frontier of Δ is not a singleton, we can always construct a game and an information structure such that the convex hull of the feasible payoffs in this extended game is Δ and for at least one player the value of information is not positive.

Remark 8. Theorem 6 would not hold with a stronger definition of PVIP requiring that every Nash equilibrium payoff of $\Gamma(G, \mathcal{S})$ dominates some Nash equilibrium payoff of $\Gamma(G, \mathcal{T})$ that is, the set of Nash set equilibrium payoffs in $\Gamma(G, \mathcal{S})$ is “above” the set of Nash equilibrium payoffs in $\Gamma(G, \mathcal{T})$. The following is a counterexample.

Let, as usual, G_A and G_B be chosen by Nature with equal probabilities, where

$$G_A = \begin{array}{|c|c|c|} \hline 50, 50 & 0, 0 & 0, 0 \\ \hline 0, 0 & 9, 9 & 0, 10 \\ \hline 0, 0 & 10, 0 & 1, 1 \\ \hline \end{array} \quad G_B = \begin{array}{|c|c|c|} \hline 50, 50 & 0, 0 & 0, 0 \\ \hline 0, 0 & 1, 1 & 10, 0 \\ \hline 0, 0 & 0, 10 & 9, 9 \\ \hline \end{array}$$

The game where both players are informed of the choice of Nature has, among others, $(1, 1)$ as Nash payoff. The Pareto frontier of this game is the singleton $(50, 50)$.

The game where no player is informed is

$$G = \begin{array}{|c|c|c|} \hline 50, 50 & 0, 0 & 0, 0 \\ \hline 0, 0 & 5, 5 & 5, 5 \\ \hline 0, 0 & 5, 5 & 5, 5 \\ \hline \end{array}$$

This game has, among others, the following mixed-strategy Nash payoff profile: $(50/11, 50/11)$. This is also the *lowest* equilibrium payoff since $50/11$ is the *maxmin* payoff for both players. Therefore the equilibrium payoff $(1, 1)$ in the game where all players are informed does not dominate any equilibrium payoff of the game where no player is informed.

4 More examples

In this section we will re-examine the examples introduced in Section 2, and provide more examples to illustrate the strength of Theorem 6. In particular we will show that the structure of the Pareto frontier of the payoff set was quite extreme in Examples 1 and 2: In one case it is never a singleton and in the other it is a singleton only when everybody is informed. Then we will provide a new example where the Pareto frontier is a singleton only when one player (but not the other) is informed. Finally we will introduce some examples that show why the result about PVIP had to be stated in terms of classes of games having the same convex hull of feasible payoffs, and does not hold for single games.

In Example 1 the Pareto frontier of the payoff set is not a singleton when both players are informed (it contains $(0.5, -4), (0, 0), (-4, 0.5)$). It is not a singleton when only one player is informed (for instance, when the row player is informed it contains $(0.5, -4), (0, 0)$). It follows that *none* of the games with additional information satisfy the sufficient condition for PVIP. In this example the value of information is always negative.

In Example 2 when both players are informed the Pareto frontier is a singleton $\{(2, 2)\}$. It is not a singleton when only one player is informed, which shows why the

value of information is positive when both players are informed, but not when only one of them is.

To complete the picture, in the next example we consider a game where, if only one player is perfectly informed, then there exists a unique Pareto payoff profile, therefore, by Theorem 6, any coarser information structure is worse. However, when both players are perfectly informed uniqueness of the Pareto payoff profiles does not hold, and information does not make the players better off. Theorem 6 states that when an extended game has a unique Pareto optimal payoff, then the equilibria of an analogous game with a coarser information structure are (weakly) dominated. The following example shows that, if uniqueness of the Pareto optimal payoff happens at an intermediate level of information disclosure, then both coarser and finer information structures can produce dominated equilibrium payoffs.

Example 9. Consider a two player game in which player 1 chooses the row, player 2 chooses the column, and the payoff matrix is one of the two with equal probabilities: In state $k = 1$ the payoff matrix is

	<i>N1</i>	<i>N2</i>	<i>F1</i>	<i>F2</i>	<i>P</i>
<i>N1</i>	0, 0	$-\gamma, -\gamma$	$-5, \lambda$	$-\gamma, -\gamma$	10, 10
<i>N2</i>	$-\gamma, -\gamma$	$-2\gamma, -2\gamma$	$-\gamma, -\gamma$	$-2\gamma, -2\gamma$	$-\gamma, -\gamma$
<i>F1</i>	$\lambda, -5$	$-\gamma, -\gamma$	0, 0	$-\gamma, -\gamma$	0, -10
<i>F2</i>	$-\gamma, -\gamma$	$-2\gamma, -2\gamma$	$-\gamma, -\gamma$	$-2\gamma, -2\gamma$	$-\gamma, -\gamma$
<i>P</i>	10, 10	$-\gamma, -\gamma$	-10, 0	$-\gamma, -\gamma$	-5, -5

and in state $k = 2$ the payoff matrix is

	<i>N1</i>	<i>N2</i>	<i>F1</i>	<i>F2</i>	<i>P</i>
<i>N1</i>	$-2\gamma, -2\gamma$	$-\gamma, -\gamma$	$-2\gamma, -2\gamma$	$-\gamma, -\gamma$	$-\gamma, -\gamma$
<i>N2</i>	$-\gamma, -\gamma$	0, 0	$-\gamma, -\gamma$	$-5, \lambda$	10, 10
<i>F1</i>	$-2\gamma, -2\gamma$	$-\gamma, -\gamma$	$-2\gamma, -2\gamma$	$-\gamma, -\gamma$	$-\gamma, -\gamma$
<i>F2</i>	$-\gamma, -\gamma$	$\lambda, -5$	$-\gamma, -\gamma$	0, 0	0, -10
<i>P</i>	$-\gamma, -\gamma$	10, 10	$-\gamma, -\gamma$	-10, 0	-5, -5

where γ is a very large parameter (e.g. 10^6), and $\lambda \in [0, 100]$.

Note that if a player is not informed of the state of nature, P (passive) is a strictly dominating action in the expected game. Hence, if no player is informed of k , the only equilibrium yields $(-5, -5)$.

Now assume that it is common knowledge that only one player, say the row player, is informed of k . It is still a dominating action for the column player to play P (even when the row player chooses different actions in the two states). Hence the best response of the row player is to play Nk (for Nice in state k) in state k . This yields an equilibrium payoff of $(10, 10)$, which is the only Pareto optimum.

Finally, assume both players are informed. Playing Nk or Fk in state $3 - k$ is dominated by any other action, hence the game reduces to:

	N	F	P
N	0, 0	-5, λ	10, 10
F	λ , -5	0, 0	0, -10
P	10, 10	-10, 0	-5, -5

where N and F stand for “play Nk (or Fk) in state k ”. We shall distinguish the cases $\lambda > 10$ and $\lambda < 10$.

If $\lambda > 10$, P is dominated by F (fight) for both players and once P has been deleted F also dominates N . The only equilibrium yields (0, 0). Notice, in particular, that the strategy profile (Nk, P) yielding the unique Pareto payoff (10, 10) with only one player fully informed, is no longer an equilibrium.

If $\lambda < 10$, the pure equilibria are (P, N) , (N, P) with payoff (10, 10) and (F, F) with payoff (0, 0). There are also mixed equilibria.

Let us now examine the shape of the Pareto frontier in these two cases: $\lambda > 10$ and $\lambda < 10$.

Figures 1 and 2 about here

Figure 1: The set of feasible payoffs in Example 9 when $\lambda < 10$

Figure 2: The set of feasible payoffs in Example 9 when $\lambda > 10$

The above example shows the extent of application for our result. The equilibrium payoffs are not monotone in the information when $\lambda > 10$. The equilibrium payoffs are (0, 0) when both players are informed, and (10, 10) when only one is. The information structure corresponding to public perfect information does not lead to a unique Pareto payoff, and this explains the non-monotonicity. On the other hand the information structure that corresponds to one-sided information leads to a unique Pareto payoff, namely (10, 10). If Ω is rich enough, it is possible to build on it some partial information structures by means of some signaling devices, with different correlations between the signal and the state. The higher the correlation, the better the information. Our theorem guarantees that any one-sided partial information structure (obtained for instance by signaling) for only one player cannot improve the equilibrium payoff with respect to the situation where one player is perfectly informed and the other one is uninformed.

On the other hand, when $\lambda < 10$, the Pareto frontier when both players are informed does reduce to a single point, even if there are multiple equilibria (in particular there is more than one equilibrium yielding the Pareto payoff). In this case, both players being informed is at least as good as any other information structure, in particular it is (weakly) better than if both players are only partially informed.

The final examples aims at showing why the characterization of Theorem 6 has to be given in terms of payoff sets.

Example 10. As in the games of Section 2, Nature chooses G_A or G_B with probability $1/2$.

$$G_A = \begin{array}{|c|c|} \hline -1, 2 & -1, 3 \\ \hline 0, 0 & 0, 0 \\ \hline 1, 1 & 1, 1 \\ \hline \end{array} \quad G_B = \begin{array}{|c|c|} \hline -1, 3 & -1, 2 \\ \hline 1, 1 & 1, 1 \\ \hline 0, 0 & 0, 0 \\ \hline \end{array}$$

Player 2 is a dummy. Under the information structure \mathcal{S} , where player 1 is informed, the only Nash payoff is $(1, 1)$. Under the information structure \mathcal{T} , where player 1 is not informed, the only Nash payoff is $(0.5, 0.5)$.

Hence the I-U matrix is

$$\begin{array}{cc} & \begin{array}{cc} \text{2-Inf} & \text{2-Uninf} \end{array} \\ \begin{array}{c} \text{1-Inf} \\ \text{1-Uninf} \end{array} & \begin{array}{|c|c|} \hline 1, 1 & 1, 1 \\ \hline 0.5, 0.5 & 0.5, 0.5 \\ \hline \end{array} \end{array}$$

Therefore the value of information under \mathcal{S} is positive even if in this case the Pareto frontier is not a singleton (it contains $(-1, 3), (1, 1)$).

Example 11. Let

$$G_A = \begin{array}{|c|c|} \hline 2, 2 & 9, 0 \\ \hline 0, 9 & 8, 8 \\ \hline \end{array} \quad G_B = \begin{array}{|c|c|} \hline 2, 2 & 0, 0 \\ \hline 0, 0 & 0, 0 \\ \hline \end{array}$$

Top and Left are dominant strategies in the state-games G_A and G_B , and also in $\frac{1}{2}G_A + \frac{1}{2}G_B$. Hence $(2, 2)$ is the only Nash payoff in all information structures (zero, one or two players informed about the state of Nature). However, the Pareto frontier is never a singleton, and in particular in the case of full information it contains the points $(5, 5), (5.5, 1)$ and $(1, 5.5)$.

Thus, the game has the PVIP but $\text{Pa}(\Delta)$ is not a singleton.

The above examples show that it is not true that having a unique Pareto feasible payoff under a certain information structure is necessary for the value of information to be positive. What is true is that for every closed convex set whose Pareto frontier is not a singleton, we can always build a game of incomplete information with that set as payoff set where the value of information is not positive.

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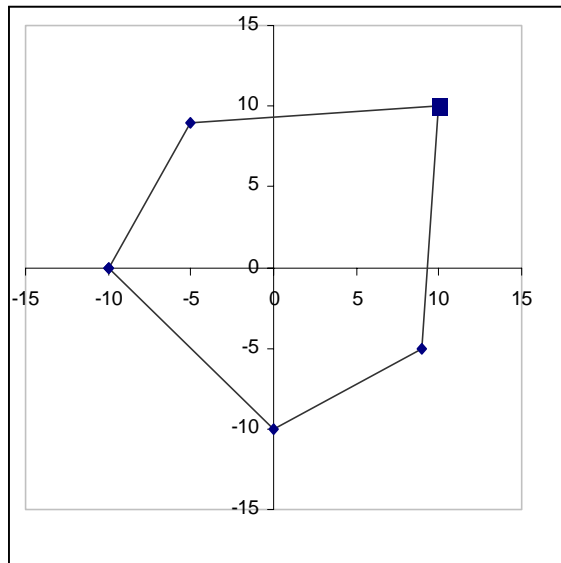


Figure 1

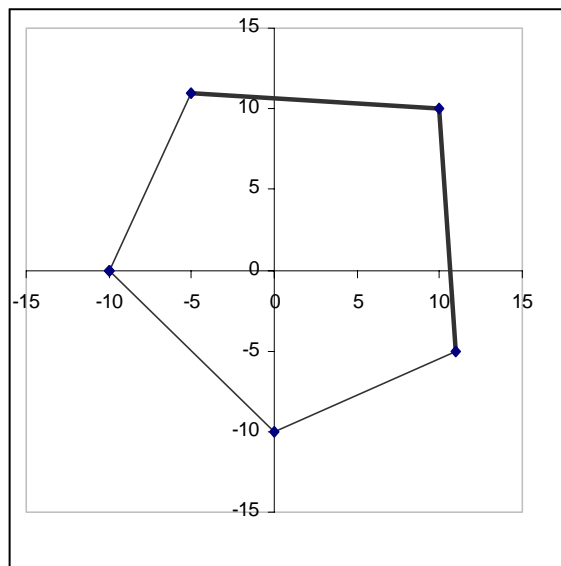


Figure 2

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