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Cores and Stable Sets of Finite Dimensional Games

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Abstract

In this paper we study exact TU games having finite dimensional non-atomic cores, a class of games that includes relevant economic games. We first characterize them by showing that they are a particular type of market games. Using this characterization, we then show that in such a class the cores are their unique von Neumann-Morgenstern stable sets.

1 Introduction

In this paper we study transferable utility (TU) games whose cores are finite dimensional subsets of $na(\Sigma)$, the space of all finitely additive non-atomic charges. This is an important class of games, which contains many relevant economic games, such as production games and games arising from exchange economies (see, e.g., Hart and Neyman, 1988, p. 32).

In an earlier paper, Marinacci and Montrucchio (2001), *inter alia* we provided simple conditions under which a game belongs to such a class and we argued that they are going to be satisfied in many cases of interest. In this paper we move on in this research line and we focus on exact games

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having finite dimensional cores in $na(\Sigma)$. For this important class of games, which we denote by $ef(\Sigma)$, we prove two main results:

- In Theorem 2 we characterize the class $ef(\Sigma)$ by showing that a game belongs to $ef(\Sigma)$ if and only if it is a generalized linear production game, a particular type of measure game whose associated function g has the very simple form $g(x) = \min_{t \in T} a(t) \cdot x$, where T is a compact metric space and $a : T \to \mathbb{R}^N$ is a continuous map.
- Using the above characterization, in Theorem 3 we show that the cores of the games in $ef(\Sigma)$ are their unique von Neumann-Morgenstern stable sets.

The interest of the first result lies in the simplicity of the characterization, as generalized linear production games have a very simple structure. Moreover, Proposition 1 will show that generalized production games are market games, and so it turns out that all elements of $ef(\Sigma)$ are indeed a special type of market games. Under a suitable continuity condition the converse holds as well, that is, suitably continuous market games are actually generalized linear production games. This is proved in Proposition 7.

The second result establishes a remarkable property of games in $ef(\Sigma)$: for them two fundamental solution concepts, cores and stable sets, are equivalent. As well-known, stable sets are in general not easy to handle, as there are typically many of them and they are not easy to find. This has greatly limited the use of stable sets, despite their conceptual appeal (see, e.g., Aumann, 1987, and Lucas, 1992). In contrast, the core is a much simpler set and it has gained a central importance in all applications. When the two solution concepts coincide we have an ideal situation, in which the conceptual appeal of stable sets and the simplicity of cores are combined.

Cases in which this "ideal" situation occurs have been discovered in some insightful papers. In particular, Shapley (1971) showed that stable sets and cores coincide for finite positive convex games, a result recently extended to infinite positive convex games by Einy and Shitovitz (1996) and Einy, Holzman, Monderer, and Shitovitz (1997). However, Proposition 5 will show that there is only a trivial overlap between the class of convex games and the class $ef(\Sigma)$.

On the other hand, Einy, Holzman, Monderer, and Shitovitz (1996) have shown that cores and stable sets coincide also for exact and positive glove market games, a special class of elements of $ef(\Sigma)$ characterized by having cores that are polytopes in $na^{+}(\Sigma)$ (see Proposition 4 below). As anticipated, in this paper we show in Theorem 3 that this equivalence result holds for all elements of the much larger class $ef(\Sigma)$; this also sheds further light on the cores of these games by showing that they feature a noteworthy property of external stability besides the usual internal one.

The paper is organized as follows. Section 2 contains some preliminary material, while Section 3 introduces closure under minorization, a notion that may be of independent interest. Sections 4 and 5, which are the heart of the paper, contain the characterization and the result on stability we just discussed. In Appendix A we collect some useful properties of m-closure, while Appendix B contains all proofs.

2 Preliminaries and Notation

2.1 Games and Cores

Throughout the paper, Ω is the set of players and Σ is the σ-algebra of admissible coalitions. Subsets of Ω are understood to be in Σ even where not stated explicitly.

A set function $\nu : \Sigma \to \mathbb{R}$ is a *game* if $\nu(\varnothing) = 0$. A game ν is

positive if $\nu(E) \geq 0$ for all E,

bounded if $\sup_{E\in\Sigma} |\nu(E)| < \infty$.

superadditive if $\nu(E) + \nu(E') \leq \nu(E \cup E')$ for all coalitions E and E' such that $E \cap E' = \varnothing$,

convex (or supermodular) if $\nu(E \cup E') + \nu(E \cap E') \geq \nu(E) + \nu(E')$ for all coalitions E and E',

additive (or a charge) if $\nu(E \cup E') = \nu(E) + \nu(E')$ for all pairwise disjoint coalitions E and E',

countably additive (or a measure) if $\nu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \nu(E_i)$ for all countable collections of pairwise disjoint coalitions $\{E_i\}_{i=1}^{\infty}$,

continuous at $A \in \Sigma$ if $\nu(A_n) \to \nu(A)$ for all the sequences $A_n \uparrow A$ and $A_n \downarrow A$, *continuous* if ν is continuous at any coalition $A \in \Sigma$.

Unless otherwise stated, charges and measures are understood to be signed. The set of all charges (measures) that are bounded with respect to the variation norm is denoted by $ba(\Sigma)$ $(ca(\Sigma))$. Generic elements of $ba(\Sigma)$ are denoted by m or μ, while its nonnegative elements are denoted by P or Q.

A charge m is *non-atomic* (or strongly continuous) if, for every $\varepsilon > 0$, there exists a finite partition $\{E_1, ..., E_n\}$ of Ω in Σ such that $|m|(E_i) \leq \varepsilon$ for all $i = 1, .., n$. When m is countable additive, it is non-atomic if and only if for all $m(E) \neq 0$ there exists $B \subseteq E$ such that $m(B) \neq 0$ and $m(E - B) \neq 0$. The set of all non-atomic charges is denoted by $na(\Sigma)$, while $na_\sigma(\Sigma) = na(\Sigma) \cap ca(\Sigma)$. We recall that $na(\Sigma)$ is a Banach sublattice of $ba(\Sigma)$.

The importance of non-atomic charges is due to their range convexity. Specifically, let $m = (m_1, ..., m_N) : \Sigma \to \mathbb{R}^N$ be a vector charge. If each m_i is non-atomic, then by the Lyapunov Theorem the range $R(m) = \{m(A) : A \in \Sigma\}$ is a convex subset of $\mathbb{R}^N$ (see Bhaskara Rao and Bhaskara Rao, 1983).

The game $\nu : \Sigma \to \mathbb{R}$ is a *measure game* if there exists $\{P_1, ..., P_N\} \subseteq na^+(\Sigma)$ and a function $g : R(P) \to \mathbb{R}$ such that

$$\nu(E) = g(P(E)) \quad \text{for all } E \in \Sigma.$$

The *core* of a game ν is

$$core(\nu) = \{m \in ba(\Sigma) : m(\Omega) = \nu(\Omega) \text{ and } m(E) \geq \nu(E) \text{ for all } E \in \Sigma\}.$$

The core of a game is a weak*-compact subset of $ba(\Sigma)$.

A game with non-empty core is called *balanced*. Given a game $\nu : \Sigma \to \mathbb{R}$, its restriction $\nu_E : \Sigma_E \to \mathbb{R}$ on a coalition E is called a *subgame*. A game whose all subgames ν_E have non-empty cores is called *totally balanced*.

The *exact envelope* ν_e of a balanced game ν is defined by $\nu_e(E) = \min_{m \in core(\nu)} m(E)$ for all $E \in \Sigma$. A balanced game ν is *exact* if $\nu = \nu_e$. Bounded convex games are an important class of exact games (see Marinacci and Montrucchio, 2001).

Given a subset $\Gamma \subseteq ba(\Sigma)$, $co(\Gamma)$ denotes its convex hull, $\overline{co}^{w^*}(\Gamma)$ its weak*-closed convex hull, and $\overline{co}^{s}(\Gamma)$ its norm-closed convex hull.

The space of all bounded Σ-measurable function on Ω, equipped with the sup norm topology, is denoted by $B(\Sigma)$. The standard duality pairing between $f \in B(\Sigma)$ and $m \in ba(\Sigma)$ will be written as $\langle f, m\rangle = \int f dm$. Finally, $B_1(\Sigma)$ denotes the elements f in $B(\Sigma)$ such that $0 \leq f \leq 1$.

2.2 Market Games

Market games will play an important role in this paper; it is a classic class of games that for example arises in modeling monetary exchange economies (see, e.g., Hart, 1977).

Their definition is based on the na-topology of $B(\Sigma)$, whose neighborhood base at $f \in B(\Sigma)$ is given by the sets $U_f(\varepsilon; m_1, ..., m_N)$ of the form:

$$\{g \in B(\Sigma) : |\langle g, m_i\rangle - \langle f, m_i\rangle| < \varepsilon \text{ for each } i = 1, ..., N\}$$

where $\{m_1, ..., m_N\} \subseteq na(\Sigma)$ and $\varepsilon > 0$. Under this topology $B(\Sigma)$ is a locally convex (not Hausdorff) topological vector space. The na-topology is the coarsest topology which makes continuous all the linear functionals $f \mapsto \langle f, m\rangle$ with $m \in na(\Sigma)$. This topology was introduced by Aumann and Shapley (1974), though the version we use here is finer than theirs, as here we allow any measure in $na(\Sigma)$ rather than just those in $na_\sigma(\Sigma)$. With the exception of Proposition 6, we will always restrict the na-topology to $B_1(\Sigma)$; that is, $B_1(\Sigma)$ will be endowed with the relative na-topology.

By the Lyapunov Theorem the characteristic functions are na-dense in $B_1(\Sigma)$. Therefore, any game $\nu : \Sigma \to \mathbb{R}$, when viewed as a function over the characteristic functions 1_E, has at most one na-continuous extension to $B_1(\Sigma)$. Following Aumann and Shapley (1974), we denote this extension by ν^*.

We can now introduce market games. A game $\nu : \Sigma \to \mathbb{R}$ is said to be a *market game* if it is superadditive and if it admits a positively homogeneous and na-continuous extension ν^* to $B_1(\Sigma)$.

This definition of market games is more general than the usual one since we are considering the na-topology rather than the coarser na_σ-topology. Nevertheless, it can be checked that market games as defined here still enjoy important properties. In particular, they are totally balanced and their extension ν^* is concave. For the latter property, notice that, by Mertens (1990), the classic extension of Lyapunov Theorem due to Dvoretzky, Wald, and Wolfowitz (1951) holds in $na(\Sigma)$ as well, and so we can still use the argument of Proposition 27.1 of Aumann and Shapley (1974) to show that ν^* is concave.

3 Closure under Minorization

To prove our characterizations we have to introduce the notion of closure under minorization, which may be of independent interest. Throughout the paper we denote by Γ_α, with $\alpha \in \mathbb{R}$, a (possibly finite) subset of $ba(\Sigma)$ such that $m(\Omega) = \alpha$ for each $m \in \Gamma_\alpha$. We will drop the subscript α whenever no confusion arises.

Definition 1 *The closure under minorization (m-closure, for short)* $\overline{\Gamma}_\alpha$ *of a set* Γ_α *is defined as follows:* $m \in \overline{\Gamma}_\alpha$ *if* $m(\Omega) = \alpha$ *and for each* $E \in \Sigma$ *there is some* $m' \in \Gamma_\alpha$ *such that* $m(E) \geq m'(E)$. *The set* Γ_α *is m-closed if* $\overline{\Gamma}_\alpha = \Gamma_\alpha$.

When Γ_α is weak*-compact, the previous definition can be equivalently formulated by using the lower envelope set function

$$\nu_{\Gamma_\alpha}(E) = \min_{m \in \Gamma_\alpha} m(E) \qquad \text{for all } E \in \Sigma,$$

which is naturally associated with Γ_α. It is immediate to check that $m \in \overline{\Gamma}_\alpha$ if and only if $m \geq \nu_{\Gamma_\alpha}$ and $m(\Omega) = \nu_{\Gamma_\alpha}(\Omega) = \alpha$.

The correspondence $\Gamma_\alpha \longmapsto \overline{\Gamma}_\alpha$ is a closure operation on the set $ba_\alpha(\Sigma) = \{m \in ba(\Sigma) : m(\Omega) = \alpha\}$, that is, it satisfies the following properties:

(i) $\Gamma_\alpha \subseteq \overline{\Gamma}_\alpha$,

(ii) $\overline{\Gamma}_\alpha \subseteq \overline{\Gamma'}_\alpha$ whenever $\Gamma_\alpha \subseteq \Gamma'_\alpha$,

(iii) $\overline{\overline{\Gamma}}_\alpha = \overline{\Gamma}_\alpha$.

The m-closure has some further useful properties, collected in Appendix A.

In order to state our first significant result, Theorem 1, we have to introduce an important map. Given a vector charge $m = (m_1, ..., m_N) : \Sigma \to \mathbb{R}^N$, let $\pi : \mathbb{R}^N \to span(R(m))$ be the orthogonal projection on the linear span $span(R(m))$. Define the map $R : span\{m_1, ..., m_N\} \to span(R(m))$ by

$$R(\chi \cdot m) = \pi(\chi) \qquad \text{for all } \chi \in \mathbb{R}^N. \tag{1}$$

The next lemma shows that R is a well-defined "canonical" isomorphism between the two key vector spaces $span\{m_1, ..., m_N\}$ and $span(R(m))$.

Lemma 1 *The map $R : span\{m_1, ..., m_N\} \to span(R(m))$ is a linear and weak*-continuous isomorphism. In particular, $\mu = R(\mu) \cdot m$ holds for all $\mu \in span\{m_1, ..., m_N\}$.*

A first application of this canonical isomorphism is given by the next result, which provides an important property of the m-closure in the finite dimensional setting.

Theorem 1 *If Γ_α is a finite dimensional subset of $na(\Sigma)$, then $\overline{\Gamma}_\alpha = co(\Gamma_\alpha)$.*

In other words, a set Γ_α is m-closed if and only if it is convex, provided $\Gamma_\alpha \subseteq na(\Sigma)$ is finite-dimensional. In order to put in sharper focus this result, in Appendix A we present few examples that show that the equality $\overline{\Gamma}_\alpha = co(\Gamma_\alpha)$ is in general false, even for finite games.

Theorem 1 delivers the following sum and difference rule for exact games (the symbol $\ominus$ denotes the Minkowski difference, defined below in Appendix A).

Corollary 1 *Let $\nu_1, \nu_2 : \Sigma \to \mathbb{R}$ be any two exact games. Then*

$$core(\nu_1 + \nu_2) = \overline{core(\nu_1) + core(\nu_2)}, \tag{2}$$

and

$$core(\nu_1 - \nu_2) = core(\nu_1) \ominus core(\nu_2). \tag{3}$$

If, in addition, $core(\nu_1)$ and $core(\nu_2)$ are both finite dimensional subsets of $na(\Sigma)$, then

$$core(\nu_1 + \nu_2) = core(\nu_1) + core(\nu_2). \tag{4}$$

Property (3) and (4) have been proved for finite and positive convex games by Danilov and Koshevoy (2000), while Marinacci and Montrucchio (2001) proved (4) for general bounded convex games. However, by Proposition 5 below there is only a trivial overlap between the class of convex games and the class of games having finite dimensional cores in $na(\Sigma)$. As a result, the derivation of property (4) for these two classes of games is altogether independent.

4 The Characterization

Eq. (4) shows that exact games having finite dimensional cores in $na\left(\Sigma\right)$ form a convex cone. Let us denote this cone by $ef\left(\Sigma\right)$. Using the properties of the m-closure, in this section we provide a general characterization of the cone $ef\left(\Sigma\right)$. In order to do this, we first have to present a class of measure games introduced by Marinacci and Montrucchio (2001).

Let $a : T \to \mathbb{R}^N$ be a continuous map defined on a compact metric space T, and define the function $g : \mathbb{R}^N \to \mathbb{R}$ by

$$g\left(x\right) = \min_{t \in T} a\left(t\right) \cdot x \qquad \text{for all } x \in \mathbb{R}^N. \tag{5}$$

We call the measure game $\nu = g\left(P\right) : \Sigma \to \mathbb{R}$ a *generalized linear production game*. When T is a finite set and $a\left(t\right) \equiv a^t \in \mathbb{R}^N_+$, we get back to the linear production games of Owen (1975) and Billera and Raanan (1981), which include glove market games.

Proposition 1 *Generalized linear production games are market games.*

In other words, generalized linear production games are a class of market games. By Marinacci and Montrucchio (2001), their cores are given by

$$core\left(\nu\right) = \left\{\chi \cdot P : \chi \in co\left(a\left(t\right) : a\left(t\right) \cdot P\left(\Omega\right) = \nu\left(\Omega\right)\right)\right\}, \tag{6}$$

a formula that generalizes earlier results for linear games of Billera and Raanan (1981).

Being market games, generalized linear production games are totally balanced. But, they are not necessarily exact. To establish when they do, for each E let

$$T\left(E\right) = \left\{t \in T : a\left(t\right) \cdot P\left(E\right) = \nu\left(E\right)\right\};$$

that is, $T\left(E\right) \subseteq T$ is the non-empty compact subset of all the minimizers at E. By (6), we then have

$$core\left(\nu\right) = \left\{\chi \cdot P : \chi \in co\left\{a\left(t\right) : t \in T\left(\Omega\right)\right\}\right\}, \tag{7}$$

and so

$$\nu_e\left(E\right) = \min_{t \in T(\Omega)} a\left(t\right) \cdot P(E).$$

This immediately leads to the following condition for exactness, trivially fulfilled when $T\left(\Omega\right) = T$.

Proposition 2 *A generalized production game is exact if and only if* $T(E) \cap T(\Omega) \neq \varnothing$ *for all* $E \in \Sigma$.

We can now state the announced characterization of games in $ef(\Sigma)$. Remarkably, it shows that an exact game has a finite-dimensional core in $na(\Sigma)$ if and only if it is an exact generalized linear production game. That is, all exact games with finite dimensional cores are market games of a particular type.

In reading the next result it is important to observe that a subset Γ of $na(\Sigma)$ is finite dimensional if and only if there exists a set $\{P_1, ..., P_N\} \subseteq na^+(\Sigma)$ such that $\Gamma \subseteq span\{P_1, ..., P_N\}$. In fact, if $\{P_1, ..., P_N\}$ were not a subset of positive non-atomic charges, we can always replace it with the "enlarged" set of positive non-atomic charges $\left\{P_1^+, P_1^-, ..., P_N^+, P_N^-\right\}$ and then use the relation $span(P_1, ..., P_N) \subseteq span\left(P_1^+, P_1^-, ..., P_N^+, P_N^-\right)$.

Theorem 2 *Let* $\{P_1, ..., P_N\} \subseteq na^+(\Sigma)$. *Given a game* $\nu : \Sigma \to \mathbb{R}$, *consider the following conditions:*

(i) ν *is exact and* $core(\nu) \subseteq span\{P_1, ..., P_N\}$,

(ii) ν *is an exact and generalized linear production game,*

(iii) there is a weak-compact set* $\Gamma_\alpha \subseteq span\{P_1, ..., P_N\}$ *such that* $\nu(E) = \min_{m \in \Gamma_\alpha} m(E)$ *for all* E,

(iv) there is a function $g : \mathbb{R}^N \to \mathbb{R}$ *concave and homogeneous of degree one such that* $\nu(E) = g(P(E))$ *for all* E.

Then,

$$(i) \iff (ii) \iff (iii) \implies (iv),$$

with

$$g(x) = \min_{\chi \in R(\Gamma_\alpha)} \chi \cdot x \qquad \text{for all } x \in \mathbb{R}^N,$$

and

$$core(\nu) = \overline{\Gamma}_\alpha = co(\Gamma_\alpha) = \{\chi \cdot P : \chi \in co(R(\Gamma_\alpha))\}. \tag{8}$$

Condition (iv) is not equivalent to the other three conditions, but it is only implied by them. As a matter of fact, if g is concave and homogeneous of degree one, the measure game $g(P)$ is totally balanced, but not necessarily exact. Next we show what is the exact envelope of $g(P)$ and, therefore, what other properties are needed on g to make the game $g(P)$ exact and condition (iv) equivalent to conditions (i)-(iii). Recall that the superdifferential at $x_0 \in \mathbb{R}^N$ of a function $g : \mathbb{R}^N \to \mathbb{R}$ is the set

$$\partial g(x_0) = \left\{ \chi \in \mathbb{R}^N : g(x) \leq g(x_0) + \chi \cdot (x - x_0) \quad \text{for all } x \in \mathbb{R}^N \right\}.$$

In particular, $\partial g(x_0) \neq \varnothing$ at all $x_0 \in \mathbb{R}^N$ provided g is concave (see, e.g., Rockafellar, 1970).

Proposition 3 *Let $g : \mathbb{R}^N \to \mathbb{R}$ be a function concave and homogeneous of degree one. Then, the measure game $\nu = g(P)$ is a totally balanced generalized linear production game, whose exact envelope is the measure game $\nu_e = g_e(P)$ with*

$$g_e(x) = \min_{\chi \in \partial g(P(\Omega))} \chi \cdot x \quad \textit{for all } x \in \mathbb{R}^N.$$

In particular, $\nu = g(P)$ is exact if and only if $g_e(x) = g(x)$ for all $x \in R(P)$.

This implies that if g is differentiable at $P(\Omega)$ – or equivalently at any point of the diagonal $tP(\Omega)$, with $t \in [0,1]$ – then $g_e(x)$ is linear and the game ν is additive.

An important class of generalized production games is given by glove market games, defined as minima of finitely many non-atomic charges. A glove market game is exact if there exists a finite set $\Gamma_\alpha = \{m_1, ..., m_N\} \subseteq na(\Sigma)$ such that

$$m(E) = \min_{m \in \Gamma_\alpha} m(E) = \min \{m_1(E), ..., m_N(E)\}$$

for all $E \in \Sigma$.[1] In view of condition (iii) of Theorem 2, these games can therefore be considered as generalized production games that are finitely generated. The next result, based on the properties of the m-closure, shows that among exact games they are characterized by having cores that are polytopes, a very special class of finite dimensional sets.[2]

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[1] Einy et al. (1996) p. 208 observe that for positive glove market games the condition is necessary as well whenever for each $t \in T$ there is E such that $T(E) = \{t\}$. By what we have discussed, this observation applies to all generalized production games. It is also easy to see that it holds when Γ consists of signed measures (see the proof of Proposition 4).

[2] A polytope is the convex hull of a finite set. It can shown that polytopes are nothing but compact polyhedra (see Aliprantis and Border, 1999).

Proposition 4 *An exact game* $\nu : \Sigma \to \mathbb{R}$ *is a glove market game if and only if its core is a polytope in* $na\,(\Sigma)$.

Example 3 in Appendix A shows that the "only if" part of this result is no longer true if in the definition of glove market games we drop the requirement that Γ_α is a subset of $na\,(\Sigma)$. In contrast, the other part would still hold.

4.1 Some Consequences

The characterization provided by Theorem 2 has some interesting consequences for the class $ef\,(\Sigma)$. The first one is given by the next Proposition, which shows that convex games cannot belong to $ef\,(\Sigma)$, unless they are additive, a trivial case from a game-theoretic standpoint. Conceptually this is an important property because it shows that the class of games which we are interested in has only a trivial overlap with the class of convex games.

Proposition 5 *A bounded convex game belongs to* $ef\,(\Sigma)$ *if and only if it is a charge.*

By Proposition 1 and Theorem 2, every game in $ef\,(\Sigma)$ is a market game. Hence, it has a positively homogeneous and *na*-continuous extension ν^* to $B_1\,(\Sigma)$. The next result provides another characterization of games in $ef\,(\Sigma)$ in terms of its extension ν^*; moreover, it relates ν^* with the (non-atomic) concave upper envelope $\overline{\nu} : B\,(\Sigma) \to \mathbb{R}$ of ν given by

$$\begin{aligned} & \overline{\nu}\,(f) \\ = \; & \inf \left\{ \langle m, f \rangle + \alpha : m \in na\,(\Sigma)\,,\, \alpha \in \mathbb{R} \text{ and } m\,(E) + \alpha \geq \nu\,(E) \;\; \forall E \in \Sigma \right\} \end{aligned}$$

for all $f \in B\,(\Sigma)$.

Proposition 6 *A game belongs to* $ef\,(\Sigma)$ *if and only if it is a market game such that its na-extension* ν^* *admits a further extension to* $B\,(\Sigma)$, *still denoted* ν^*, *having the following properties:*

(i) ν^* *is concave and positively homogeneous,*

(ii) $\nu^*\,(\alpha 1_\Omega) = \alpha \nu^*\,(1_\Omega)$, *for all* $\alpha \in \mathbb{R}$,

(iii) the function $\nu^* : B\,(\Sigma) \to \mathbb{R}$ *is na-lower semicontinuous at* 0.

In such a case, we have

$$\nu^*(f) = \overline{\nu}(f) = \min_{m \in core(\nu)} \langle m, f \rangle . \tag{9}$$

for all $f \in B_1(\Sigma)$.

Inspection of the proof of Proposition 6 shows that if we drop assumption (ii), then conditions (i) and (iii) together characterize those market games which are generalized linear production games. We thus have the following characterization of generalized linear production games that are not necessarily exact.[3]

Proposition 7 *A game is a generalized linear production game if and only if it is a market game such that its na-extension* ν^* *admits a further extension to* $B(\Sigma)$ *which is concave, positively homogeneous, and na-lower semicontinuous at* 0.

5 Stability

We now present an application of our theory by proving that the cores of all continuous games in $ef(\Sigma)$ are their unique von Neumann-Morgenstern stable sets.

In order to do this, we need some terminology. Let

$$I(\nu) = \{m \in ba(\Sigma) : m(\Omega) \leq \nu(\Omega)\}$$

be the set of all (sub)imputations. In our non-atomic setting we prefer not to require any individual rationality condition on the set $I(\nu)$ since our results do not need it and, of course, they would continue to hold under any notion of individual rationality one would like to impose on $I(\nu)$.

A coalition E is *null* is $\nu(E \cup F) = \nu(F)$ for all F such that $F \cap E = \varnothing$. Given $\xi, \eta \in I(\nu)$ and $A \in \Sigma$, we say that η *dominates* ξ *via* A, written $\eta \succ_A \xi$, if $\xi(A) < \eta(A) \leq \nu(A)$, $\eta(E) \geq \xi(E)$ for all $E \subseteq A$, and $\eta(E) > \xi(E)$ for all non-null subcoalitions $E \subseteq A$.

A subset V of $I(\nu)$ is a *(von Neumann-Morgenstern) stable set* if it satisfies the following two conditions:

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[3] We omit the proof since it is similar to that of Proposition 6.

(i) Internal stability: if $\xi \in V$, there is no $A \in \Sigma$ and $\eta \in V$ such that $\eta \succ_A \xi$.

(ii) External stability: if $\xi \in I(\nu) \backslash V$, there is some $A \in \Sigma$ and $\eta \in V$ such that $\eta \succ_A \xi$.

We can now state our result.

Theorem 3 *Let ν be a continuous game in $ef(\Sigma)$. Then the core of ν is its unique stable set.*

The proof of the result rests on the following lemma of independent interest. Einy et al. (1997) proved a similar lemma when $S = \Omega$ for continuous and positive convex games. Here, using different techniques based on Theorem 2, we show that it holds for continuous games in $ef(\Sigma)$.

Lemma 2 *Let $\nu \in ef(\Sigma)$ be a continuous game and S a fixed coalition in Σ. For all $\xi \in ca(\Sigma)$, there exist $A \subseteq S$ and $\eta \in core(\nu)$ such that*

$$\begin{aligned} \nu(A) - \xi(A) &= \max_{C \subseteq S} \{\nu(C) - \xi(C)\} \qquad (10) \\ \eta(A) &= \nu(A) \ \textit{and} \ \eta(B) \geq \xi(B) \ \textit{for all} \ B \subseteq A. \end{aligned}$$

A Appendix: More on Closure under Minorization

In this Appendix we study in more detail closure under minorization. We begin with few properties, collected in few lemmas for later use in Appendix B. We will focus on the weak*-compact case as it is the most important case for our purposes.

Lemma 3 *Suppose Γ_α is a weak*-compact set. Then:*

(i) $\overline{\Gamma}_\alpha$ is weak-compact and convex, and $\overline{co}^{w^*}(\Gamma_\alpha) \subseteq \overline{\Gamma}_\alpha$.*

(ii) ν_{Γ_α} is the unique exact game such that $core(\nu) = \overline{\Gamma}_\alpha$; in particular,

$$\nu_{\Gamma_\alpha}(E) = \min_{m \in \overline{\Gamma}_\alpha} m(E) \qquad \textit{for all } E \in \Sigma.$$

Lemma 3 has some noteworthy consequences. First, a weak*-compact set Γ_α is the core of a game if and only if it is m-closed. For, cores are clearly m-closed, while, by point (ii) of Lemma 3, the m-closed sets $\overline{\Gamma}_\alpha$ are cores of the games $\nu_{\overline{\Gamma}_\alpha}$.

Second, by point (ii) there exists a one-to-one correspondence between exact games and m-closed weak*-compact sets, given by

$$\Gamma_\alpha \longmapsto \nu_{\Gamma_\alpha},$$

with $\Gamma_\alpha = \overline{\Gamma}_\alpha$. This correspondence closely parallels the one-to-one relation existing between closed convex sets and support functions.

The next lemma, based on Schmeidler (1972), shows what happens when Γ_α is a subset of $ca\,(\Sigma)$.

Lemma 4 *Suppose Γ_α is a weak*-compact set. Then, $\Gamma_\alpha \subseteq ca\,(\Sigma)$ if and only if the lower envelope game ν_{Γ_α} is continuous at $\varnothing$ and Ω. In such a case, we have:*

(i) $\overline{\Gamma}_\alpha$ is a weakly compact subset of $ca\,(\Sigma)$ and ν_{Γ_α} is continuous.

(ii) $\overline{\Gamma}_\alpha \subseteq L^1(\Sigma, \lambda)$ for some $\lambda \in ca^+(\Sigma)$; in particular, $\lambda \in \overline{co}^s(\Gamma_\alpha)$ if $\Gamma_\alpha \subseteq ca^+(\Sigma)$.

(iii) $\overline{\Gamma}_\alpha \subseteq na\,(\Sigma)$ if $\Gamma_\alpha \subseteq na\,(\Sigma)$.

Remark. By (ii), $\overline{\Gamma}$ can be isometrically imbedded into $L^1(\Sigma, \lambda)$ by the map $m \to dm/d\lambda$, which is its Radon-Nikodym derivative.

The final lemma deals with the behavior of the m-closure with respect to the Minkowski difference of sets. Given two weak*-compact sets Γ_α and G_β in $ba\,(\Sigma)$, their *Minkowski difference* is the (possibly empty) set $\Gamma_\alpha \ominus G_\beta$ defined as

$$\Gamma_\alpha \ominus G_\beta = \{m \in ba\,(\Sigma) : m + G_\beta \subseteq \Gamma_\alpha\}.$$

The set $\Gamma_\alpha \ominus G_\beta$ is the maximal set $\Delta_{\alpha-\beta}$ such that $\Delta + G_\beta \subseteq \Gamma_\alpha$. This operation is also known as star difference (see, e.g., Penot, 1985).

Lemma 5 *Let Γ_α and G_β be weak*-compact sets of $ba\,(\Sigma)$. Then:*

(i) $\Gamma_\alpha \ominus G_\beta$ is weak-compact,*

(ii) *if* Γ_α *is* m*-closed, then* $\Gamma_\alpha \ominus G_\beta$ *is* m*-closed,*

(iii) *the game* $\nu_{\Gamma_\alpha} - \nu_{G_\beta}$ *has non-empty core if and only if* $\overline{\Gamma}_\alpha \ominus G_\beta \neq \varnothing$*; in this case*

$$\mathit{core}\left(\nu_{\Gamma_\alpha} - \nu_{G_\beta}\right) = \overline{\Gamma}_\alpha \ominus G_\beta.$$

Next we report the examples illustrating Theorem 1.

Example 1. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and $\Sigma = 2^\Omega$. Consider the compact and convex set of probability charges

$$\Gamma_1 = \left\{P \in ba^+(\Sigma) : P(\Omega) = 1 \text{ and } P(\omega_1) \leq P(\omega_2)\right\}.$$

It is easy to check that the charge P such that $P(\omega_1) = P(\omega_3) = 1/2$ belongs to $\overline{\Gamma}_1$, but it does not belong to $co(\Gamma_1)$. ▲

Example 2. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and $\Sigma = 2^\Omega$. Set $\Gamma_1 = \{P, Q\}$, where P and Q are the probability charges such that $P(\omega_1) = P(\omega_2) = 1/2$ and $Q(\omega_3) = 1$, respectively. It is easy to see that:

$$\overline{\Gamma}_1 = \{(x_1, x_2, x_3) \in [0, 1/2] \times [0, 1/2] \times [0, 1] : x_1 + x_2 + x_3 = 1\}.$$

Hence, $\overline{\Gamma}_1$ is larger than $co\{P, Q\}$. Since the game ν_{Γ_1} generated by Γ_1 is convex, this example, based on Wasserman and Kadane (1990), shows that Theorem 1 fails even for sets of measures generating convex games. ▲

Example 3. Let $\Omega = [0, 1]$ and Σ is its Borel σ-algebra. Set $\Gamma_1 = \{\delta_0, \lambda\}$, where δ_0 is the Dirac measure concentrated at 0 and λ is the Lebesgue measure on $[0, 1]$. By Lemma 3, the m-closure $\overline{\Gamma}_1$ is given by the core of the continuous and convex game

$$\nu_{\Gamma_1}(E) = \min\{\delta_0(E), \lambda(E)\} \qquad \text{for all } E \in \Sigma.$$

As

$$\nu(E) = \begin{cases} \lambda(E) & \text{if } 0 \in E, \\ 0 & \text{if } 0 \notin E \end{cases}$$

a probability measure P belongs to $\overline{\Gamma}_1$ if and only if $P(E) \geq \lambda(E)$ for all E containing 0. By Lemma 4, $\mathit{core}(\nu_{\Gamma_1}) \subseteq ca(\Sigma)$. It is easy to check that

$$\mathit{core}(\nu_{\Gamma_1}) = \overline{\Gamma}_1 = \left\{\left(1 - \int f d\lambda\right)\delta_0 + f d\lambda : 0 \leq f \leq 1 \ \lambda\text{-a.e.}\right\}.$$

Clearly, $\overline{\Gamma}_1$ is not finite dimensional and it is therefore much larger than the finite dimensional set $co(\Gamma_1)$. ▲

B Appendix: Proofs

Lemma 1. We first show that R is well defined. Let $\chi, \chi' \in \mathbb{R}^N$ be such that $\chi \cdot m = \chi' \cdot m$. Then, $(\chi - \chi') \cdot m(E) = 0$ for all $E \in \Sigma$, and so $(\chi - \chi') \cdot m \in span\,(R(m))^{\perp}$. Hence, $\pi(\chi) - \pi(\chi') = \pi(\chi - \chi') = 0$, which implies $\pi(\chi) = \pi(\chi')$. It is easy to check that R is a linear isomorphism. Moreover, $span\,\{m_1, ..., m_N\}$ with the relative weak*-topology is a finite-dimensional topological vector space. Hence, R is weak*-continuous. ■

Theorem 1. Let $P = (P_1, ..., P_N) : \Sigma \to \mathbb{R}^N_+$ be a non-atomic positive charge. Assume first that $\Gamma_\alpha \subseteq span\,\{P_1, P_2, ..., P_N\}$. We can write $m = R(m) \cdot P$ for each $m \in \Gamma_\alpha$. Consider the function $g(x) = \min_{m \in \Gamma_\alpha} R(m) \cdot x$ for all $x \in \mathbb{R}^N$ and the associated measure game $\nu = g(P)$, defined by

$$\nu(E) = \min_{m \in \Gamma_\alpha} R(m) \cdot P(E) = \min_{m \in \Gamma_\alpha} m(E) \qquad \text{for all } E \in \Sigma.$$

Clearly, ν is exact and $core\,(\nu) = \overline{\Gamma}_\alpha$ by Lemma 3.

On the other hand, the canonical map R is continuous on the weak*-compact set Γ_α, which is metrizable since it is a finite-dimensional subset of $ba\,(\Omega)$. Hence, it is a generalized linear production game. Therefore, according to (6), we have

$$\begin{aligned} core\,(\nu) &= \{\chi \cdot P : \chi \in co\,(R(m) : R(m) \cdot P(\Omega) = \nu(\Omega)) = \alpha\} \\ &= \{\chi \cdot P : \chi \in co\,(R(m) : m \in \Gamma_\alpha)\} = co\,(R(m) \cdot P : m \in \Gamma_\alpha) \\ &= co\,(\Gamma_\alpha). \end{aligned}$$

We conclude that $\overline{\Gamma}_\alpha = co\,(\Gamma_\alpha)$, as desired.

Next suppose $\Gamma_\alpha \subseteq span\,\{m_1, m_2, ..., m_N\}$, where $m = (m_1, ..., m_N) : \Sigma \to \mathbb{R}^N$ is a not necessarily positive vector charge. Since

$$\Gamma_\alpha \subseteq span\,\left\{m_1^+, m_2^+, ..., m_N^+, m_1^-, m_2^-, ..., m_N^-\right\},$$

with all $m_i^+, m_i^- \in na_+(\Sigma)$, by what established before we conclude that $\Gamma_\alpha = co\,(\Gamma_\alpha) = \overline{\Gamma}_\alpha$. ■

Corollary 1. Suppose ν_1 and ν_2 be exact. We have

$$\begin{aligned} (\nu_1 + \nu_2)(A) &= \nu_1(A) + \nu_2(A) = \min_{m_1 \in core(\nu_1)} m_1(A) + \min_{m_2 \in core(\nu_2)} m_2(A) \\ &= \min_{m \in core(\nu_1) + core(\nu_2)} m(A). \end{aligned}$$

By (ii) of Lemma 3, we have

$$core\,(\nu_1+\nu_2)=\overline{core\,(\nu_1)+core\,(\nu_2)}.$$

As to the difference rule, it follows from point (iii) Lemma 5 by taking $\Gamma = core\,(\nu_1)$ and $G = core\,(\nu_2)$.

As to the second part, it suffices to observe that $core\,(\nu_1) + core\,(\nu_2)$ is a convex and finitely dimensional subset of $na\,(\Sigma)$. By Theorem 1,

$$\overline{core\,(\nu_1)+core\,(\nu_2)}=core\,(\nu_1)+core\,(\nu_2)\,,$$

as desired ■

Proposition 1. Let $\nu : \Sigma \to \mathbb{R}$ be a generalized linear production game. By definition, there exists a superlinear function $g : \mathbb{R}^N \to \mathbb{R}$, given by (5), such that $\nu = g\,(P)$. This implies that ν is superadditive. Moreover, being concave, the function g is uniformly continuous on $R\,(P)$; hence, for all $\varepsilon > 0$ there is $\delta > 0$ such that

$$\|P\,(E)-P\,(E')\|<\delta \Longrightarrow |g\,(P\,(E))-g\,(P\,(E'))|=|\nu\,(E)-\nu\,(E')|<\varepsilon.$$

The game ν is therefore na-uniformly continuous and so it admits a unique continuous extension ν^* to $B_1\,(\Sigma)$. Since the concave and linearly homogeneous function $g\left(\int f dP\right)$, defined over $B\,(\Sigma)$, is na-continuous, we then have $\nu^*\,(f) = g\left(\int f dP\right)$ for all $f \in B_1\,(\Sigma)$. Hence, ν^* is positively homogeneous and so ν is a market game. ■

Theorem 2. In view of (6), (ii) and (iii) are equivalent. On the other hand, if we set $\Gamma_\alpha = core\,(\nu)$, clearly (i) implies (iii). As to the converse, $\nu = \min_{m\in\Gamma_\alpha} m$ implies $\nu = \min_{m\in\overline{\Gamma}_\alpha} m$. By Lemma 3, $core\,(\nu) = \overline{\Gamma}_\alpha$. By Proposition 1, $core\,(\nu) = co\,(\Gamma_\alpha)$ and so $core\,(\nu) \subseteq span\,\{P_1, ..., P_N\}$.

To prove that (iii) implies (iv), set $g\,(x) = \min_{\chi\in R(\Gamma_\alpha)} \chi \cdot x$ for all $x \in \mathbb{R}^N$, where R is given by (1). The function $g : \mathbb{R}^N \to \mathbb{R}$ is well defined because $R\,(\Gamma_\alpha)$ is a compact subset of $\mathbb{R}^N$. Clearly, it is also concave and homogeneous of degree one. Moreover, we have

$$\nu\,(E)=\min_{m\in\Gamma_\alpha} m\,(E)=\min_{m\in\Gamma_\alpha} R\,(m)\cdot P\,(E)=\min_{\chi\in R(\Gamma_\alpha)} \chi\cdot P\,(E)=g\,(P\,(E))\,,$$

which proves that (iii) implies (iv). As Theorem 1 immediately implies Eq. (8), the proof of the Theorem is completed. ■

Proposition 3. Let $\nu = g(P)$, where g is a finite concave and linearly homogeneous function on $\mathbb{R}^n$. It is well-known (see Section 13 in [24]) that g is a support function, i.e., is the Fenchel conjugate of the indicator function of some compact and convex set C. By using notation from [24], $g = \delta^*(.;C)$. Moreover, $C = \partial g(0) = \{\chi : \chi \cdot x \geq g(x) \text{ for all } x \in \mathbb{R}^n\}$. Clearly, from

$$g(x) = \min_{\chi \in C} \chi \cdot x,$$

we infer that $\nu = g(P)$ is a generalized linear production game. By (6), the core of $\nu = g(P)$ is

$$\begin{aligned} \mathit{core}(\nu) &= \{\chi \cdot P : \chi \in C \text{ and } \chi \cdot P(\Omega) = g(P(\Omega))\} \\ &= \{\chi \cdot P : \chi \in \partial g\}. \end{aligned}$$

In view of the definition of the concave function g_e, we have

$$g_e(P(A)) = \min_{\chi \in \partial g(P(\Omega))} \chi \cdot P(A) = \min_{m \in \mathit{core}(\nu)} m(A).$$

Hence, $g_e(P)$ is the exact envelope of $g(P)$. All the other claims are obvious. ■

Proposition 4. Let ν be an exact game whose core is a polytope. There exists a finite subset $\Gamma = \{m_1, ..., m_n\} \subseteq \mathit{core}(\nu)$ consisting of the extremal points of $\mathit{core}(\nu)$ and $\mathit{co}(\Gamma) = \mathit{core}(\nu)$ (see Lemma 5.123 of [1]). By Lemma 3, $\Gamma \subseteq \mathit{co}(\Gamma) \subseteq \overline{\Gamma}$ implies $\overline{\Gamma} \subseteq \overline{\mathit{co}(\Gamma)} \subseteq \overline{\Gamma}$, namely, $\overline{\Gamma} = \mathit{core}(\nu)$. Therefore,

$$\min_{m \in \Gamma} m(E) = \min_{m \in \mathit{core}(\nu)} m(E) = \nu(E),$$

and so ν is a glove market game.

Conversely, suppose that ν is an exact glove market game relative to a set $\{m_1, ..., m_n\} \subseteq na(\Sigma)$. There exists $m \in na(\Sigma)$ such that $m_i \geq m$ for each $i = 1, ..., n$. The game $\nu - m$ is a positive exact market game. By what proved on p. 208 of [11] for positive exact glove market game, it holds $(m_i - m)(\Omega) = (\nu - m)(\Omega)$ for each $i = 1, ..., n$. Hence, for $\alpha = \nu(\Omega)$, we have $\{m_1, ..., m_n\} \subseteq \Gamma_\alpha$. By Lemma 3, $\mathit{core}(\nu) = \overline{\Gamma}_\alpha$ and so, by Theorem 1, $\mathit{core}(\nu) = \mathit{co}(\Gamma_\alpha)$. Hence, $\mathit{core}(\nu)$ is a polytope. ■

Proposition 5. Let ν be a bounded convex game in $ef(\Sigma)$. By Theorem 2, there is $\{P_1, ..., P_N\} \subseteq na^+(\Sigma)$ and $g : \mathbb{R}^N \to \mathbb{R}$ such that $\nu = g(P)$ is a generalized production game. Let $E \in \Sigma$ be such that $P(E) \geq (1/2)\, P(\Omega)$. We want to show that E is linear, i.e., $\nu(E) + \nu(E^c) = \nu(\Omega)$. As $P(E) \geq (1/2)\, P(\Omega) \geq P(E^c)$, by the Lyapunov Theorem there exists $F \subseteq E$ such that $P(F) = P(E^c)$. Since ν is convex, there is $m \in core(\nu)$ such that $m(E) = \nu(E)$ and $m(F) = \nu(F)$. As $core(\nu) \subseteq span(P_1, ..., P_N)$, for some $\chi \in \mathbb{R}^N$ we have $\nu(E) = \chi \cdot P(E)$ and $\nu(F) = \chi \cdot P(F)$. Therefore,

$$\nu(E) + \nu(E^c) = \nu(E) + \nu(F) = \chi \cdot (P(E) + P(F)) = \chi \cdot P(\Omega) = \nu(\Omega),$$

and so E is a linear coalition. This implies that $m(E) = \nu(E)$ for all $m \in core(\nu)$. The same argument shows that if $P(E) \leq (1/2)\, P(\Omega)$, then E is linear.

Now let A be such that $P(A) = (1/2)\, P(\Omega)$. Clearly, A is linear. Given any $E \in \Sigma$, we have $P(E \cup A) \geq (1/2)\, P(\Omega) \geq P(E \cap A)$. By what has been just proved, both $E \cup A$ and $E \cap A$ are linear. Hence, for all $m, m' \in core(\nu)$ we have

$$\begin{aligned} m(E) &= m(E \cup A) + m(E \cap A) - m(A) \\ &= m'(E \cup A) + m'(E \cap A) - m'(A) = m'(E). \end{aligned}$$

As this holds for all $E \in \Sigma$, $core(\nu)$ is a singleton, say $core(\nu) = \{m\}$. As ν exact, we then have $\nu = m$, as desired. ■

Proposition 6. "Only if part". Assume $\nu \in ef(\Sigma)$. By Theorem 2, there exists a positively homogeneous concave function $g : \mathbb{R}^N \to \mathbb{R}$ such that $\nu = g(P)$. The same argument, adopted in Proposition 1, shows that ν admits the unique *na*-continuous extension ν^* to $B_1(\Sigma)$, given by $g\left(\int f dP\right)$. Let us prove that (9) holds. We have:

$$\begin{aligned} \nu^*(f) &= g\left(\int f dP\right) = \min_{\chi \in R(core(\nu))} \chi \cdot \int f dP \qquad (11) \\ &= \min_{\chi \in R(core(\nu))} \langle f, \chi \cdot P \rangle = \min_{m \in core(\nu)} \langle f, m \rangle. \end{aligned}$$

It remains to check that $\overline{\nu} = \nu^*$ on $B_1(\Sigma)$. Let us first prove that $\overline{\nu} \geq \nu^*$. Suppose *per contra* that $\overline{\nu}(f) < \nu^*(f)$ for some $f \in B_1(\Sigma)$. Then there exists $m \in na(\Sigma)$ and $\alpha \in \mathbb{R}$ such that $\langle m, f \rangle + \alpha < \nu^*(f)$ and $m(E) + \alpha \geq \nu(E)$ for all $E \in \Sigma$. Since both m and ν^* are *na*-continuous, there exists an *na*-neighborhood U_f of f such that $\langle m, \varphi \rangle + \alpha < \nu^*(\varphi)$ for all $\varphi \in U_f$. By the na-density of the characteristic functions, there is some E such that $m(E) + \alpha < \nu(E)$, a contradiction.

As to the opposite inequality, let $f \in B_1(\Sigma)$. As $\nu^*(f) = \min_{m \in core(\nu)} \langle f, m \rangle$, there is $\overline{m} \in core(\nu)$ such that $\nu^*(f) = \langle \overline{m}, f \rangle$. Consequently, $\overline{\nu}(f) \leq \langle \overline{m}, f \rangle = \nu^*(f)$, as desired.

Clearly, ν^* admits an extension to $B(\Sigma)$, given by the support function $\nu_e = \min_{m \in core(\nu)} \langle m, f \rangle$ which satisfies trivially conditions (i) and (ii). To end the proof, we show that indeed $\nu_e(f)$ is na-continuous. In view of (11), we have $\nu_e(f) = g\left(\int f dP\right)$. But g is uniformly continuous over $\mathbb{R}^N$ (actually it is globally Lipschitz on $\mathbb{R}^N$). Clearly, this implies that ν_e is na-continuous.

"If part". Assume that ν is an exact market game. Denote by ν^* its extension to $B(\Sigma)$, as claimed by the Proposition. As the na-topology in $B(\Sigma)$ is weaker than the $\sigma(B(\Sigma), ba(\Sigma))$-topology, ν^* turns out to be $\sigma(B(\Sigma), ba(\Sigma))$-lower semicontinuous at 0. By the well-known global property of concave functions, ν^* is $\sigma(B(\Sigma), ba(\Sigma))$-continuous on $B(\Sigma)$ (see, for instance, Thm. 5.27 of [1]). By Hormander's theorem [16], ν^* is a support function and there exists a unique weak*-compact and convex set $\Gamma \subset ba(\Sigma)$ such that

$$\nu^*(f) = \min_{m \in \Gamma} \langle m, f \rangle . \tag{12}$$

and Γ is of finite dimension (see Th. 6 of [16]). We want to prove that actually $\Gamma \subset na(\Sigma)$. To this purpose we follow the argument of [16] p. 184.

Since the sets

$$\{f \in B(\Sigma) : |\langle m_i, f \rangle| < \varepsilon\}$$

with $\{m_i\}_{i=1}^n \subseteq na^1(\Sigma)$ form an na-neighborhood base at 0, also the sets $\{f \in B(\Sigma) : |\langle m_i, f \rangle| < 1\}$, with $\{m_i\}_{i=1}^n \subseteq na(\Sigma)$ form an na-neighborhood base at 0. Consider the na-neighborhood

$$V(m_1, ..., m_n) = \{f \in B(\Sigma) : |\langle m_i, f \rangle| < 1\}$$

of 0. Let K be the convex hull of the $2n$ charges $\pm m_i$. Set

$$h_K(f) = \min_{m \in K} \langle m, f \rangle$$

for all $f \in B(\Sigma)$. It is easy to check that $h_K(f) = \min_{i=1,...,n}(-|\langle m_i, f \rangle|) = -\max_{i=1,...,n} |\langle m_i, f \rangle|$. Hence, $V(m_1, ..., m_n) = \{f \in B(\Sigma) : h_K(f) > -1\}$.

By the na-lower continuity of ν^* at 0, there exists an na-neighborhood $V(m_1, ..., m_n) = \{f \in B(\Sigma) : h_K(f) > -1\}$ of 0 such that

$$h_K(f) > -1 \Longrightarrow \nu^*(f) > -1. \tag{13}$$

In turn, this implies that $\nu^* (f) \geq h_K (f)$ for all $f \in B(\Sigma)$. For, suppose there exists $f \in B(\Sigma)$ such that $\nu^* (f) < h_K (f)$, i.e., $-\nu^* (f) > -h_K (f) \geq 0$. Then, there would exist a positive scalar α such that $-\alpha\nu^* (f) > 1 > -\alpha h_K (f)$. As both the functions ν^* and h_K are positively homogeneous, we infer that $\nu^* (\alpha f) < -1$ and $h_K (\alpha f) > -1$, contradicting (13).

We conclude that $\nu^* (f) \geq h_K (f)$ for all $f \in B(\Sigma)$. Hence, $\Gamma \subseteq K$, and so it is a finite dimensional subset of $na(\Sigma)$.

Setting $f = 1_\Omega$ and $f = -1_\Omega$ in Eq. (12), and exploiting condition (ii) we get easily that $m(\Omega) = \nu(\Omega)$ for all $m \in \Gamma$. To complete the proof, by (12) for all $E \in \Sigma$ we have

$$\nu(E) = \min_{m \in \Gamma} m(E)$$

Hence, the game is exact. Moreover, as Γ is a finite dimensional subset of $na(\Sigma)$, Theorem 1 implies that $core(\nu) = co(\Gamma) = \Gamma$. Consequently, ν is an exact game in $ef(\Sigma)$. ■

Lemma 2. According to Lemma 4, as ν is continuous, $core(\nu) \subseteq span\{P_1, ..., P_N\}$ with each $P_i \in na_\sigma^+(\Sigma)$. By Theorem 2, there is a concave and positively homogeneous $g : \mathbb{R}^N \to \mathbb{R}$ such that $\nu = g(P)$. Set $P^* = \sum_{i=1}^n P_i$. Clearly, $core(\nu) \subseteq L_1(P^*)$.

To prove this Lemma, we need some notation. Throughout the proof we fix a coalition S. We denote by ν_S the subgame of ν on Σ_S, the restriction of Σ to S. For any $m \in ba(\Sigma)$, $m_S \in ba(\Sigma_S)$ is its restriction to S. Moreover, $L : B(\Sigma_S) \to B(\Sigma)$ is the natural extension defined by

$$L(\varphi)(\omega) = \begin{cases} \varphi(\omega) & \text{if } \omega \in S \\ 0 & \text{if } \omega \notin S \end{cases}$$

Its adjoint operator $L^* : ba(\Sigma) \to ba(\Sigma_S)$ is defined by $L^*(m) = m_S$ for each $m \in ba(\Sigma)$. If ν^* is the na-continuous extension of ν, ν_S^* denotes its restriction to $B_1(\Sigma_S)$, i.e., $\nu_S^* = \nu^* \circ L$. Finally, $\nu_e : B(\Sigma) \to \mathbb{R}$ denotes the support function $\nu_e(f) = \min_{m \in core(\nu)} \langle m, f \rangle$.

By Proposition 6, we have:

$$\nu_S^*(\varphi) = g\left(\int_S \varphi dP\right) = (\nu_e \circ L)(\varphi) \qquad (14)$$

for all $\varphi \in B_1(\Sigma_S)$, and where $\int_S \varphi dP = \left(\int_S \varphi dP_i\right)_{i=1}^N$.

Finally, let $U = \{\varphi \in L_\infty(P_S^*) : 0 \leq \varphi \leq 1\ P_S^* - a.e.\}$. Clearly, ν_S^* is well defined on U by setting $\nu_S^*(\varphi) = g\left(\int_S \varphi dP\right)$ for all $\varphi \in U$.

We begin the proof by assuming that $\xi \in L_1(P^*)$. Consider the function $\nu_S^* - \xi_S : U \to \mathbb{R}$. If $P^*(S) = 0$, this function is equal to zero on U and so it trivially achieves its maximum. Assume that $P^*(S) > 0$. The function $\xi_S : U \to \mathbb{R}$ is $\sigma(L_\infty, L_1)$-continuous because $\xi_S \in L^1(P_S^*)$. The same is true for ν_S^*. In fact, for all $\varphi \in U$ we have:

$$\nu_S^*(\varphi) = g\left(\int_S \varphi dP_1, ..., \int_S \varphi dP_N\right) = g\left(\int_S \varphi \frac{dP_1}{dP^*} dP^*, ..., \int_S \varphi \frac{dP_N}{dP^*} dP^*\right).$$

As g is continuous and each dP_i/dP^* belongs to $L^1(P^*)$, the function $\nu_S^* : U \to \mathbb{R}$ is $\sigma(L_\infty, L_1)$-continuous. Therefore, $\nu_S^* - \xi_S : U \to \mathbb{R}$ achieves its maximum on U by the Alaoglu Theorem. This maximum is attained at some 1_A. In fact, let $\overline{\varphi} \in \arg\max_{\varphi \in U} \nu_S^*(\varphi) - \langle \xi_S, \varphi \rangle$. Consider the vector measure $\left(P_1, P_2, ..., P_n, \xi^+, \xi^-\right)_S$. As $\xi \in L^1(P^*)$ is a non-atomic signed measure, by the Lyapunov Theorem there exists some $A \subseteq S$ such that $P_i(A) = \int_S \overline{\varphi} dP_i$ for each i, $\xi^+(A) = \int_S \overline{\varphi} d\xi^+$, and $\xi^-(A) = \int_S \overline{\varphi} d\xi^-$. Thus, $\nu_S^*(\overline{\varphi}) - \langle \xi_S, \overline{\varphi} \rangle = \nu_S^*(1_A) - \langle \xi_S, 1_A \rangle$ and so the maximum is attained at 1_A.

This implies that

$$\nu_S^*(\varphi) \leq \nu(A) + \langle \xi_S, \varphi \rangle - \xi_S(A)$$

for all $\varphi \in U$. In particular, this is the case for all $\varphi \in B_1(\Sigma_S)$ and so $\xi_S \in \partial \nu_S^*(1_A)$, where $\partial \nu_S^*(1_A)$ denotes the superdifferential of the concave function ν_S^* defined on $B_1(\Sigma_S)$. Now, ν_S^* can be viewed as the sum of two functions defined over $B(\Sigma_S)$. Actually, by (14) we have $\nu_S^* = (\nu_e \circ L) + I$, where

$$I(\varphi) = \begin{cases} 0 & \text{if } \varphi \in B_1(\Sigma_S) \\ -\infty & \text{if } \varphi \notin B_1(\Sigma_S) \end{cases}$$

By the sum rule for superdifferentials (see, e.g., Thm. 3.2.6 of [4]), we have

$$\partial \nu_S^*(1_A) = \partial(\nu_e \circ L)(1_A) + \partial I(1_A).$$

It holds $\partial I(1_A) = K_A^S$, where

$$K_A^S = \{m \in ba(\Sigma_S) : m(F) \geq 0,\ m(G) \leq 0 \text{ for all } F \subseteq A^c \text{ and } G \subseteq A\}.$$

Moreover, ν_e being continuous, we have

$$\partial(\nu_e \circ L)(1_A) = L^* \partial(\nu_e)(L 1_A) = L^* \partial(\nu_e)(1_A)$$

(see, e.g., Thm. 1.5 of [23]). Since,

$$\partial\left(\nu_e\right)\left(1_A\right)=\left\{m\in core\left(\nu\right):m\left(A\right)=\nu\left(A\right)\right\}$$

(see Lemma 3 of [19]), we then have

$$\xi_S\in\partial\nu_S^*\left(1_A\right)=L^*\partial\nu_e\left(1_A\right)+K_A^S. \tag{15}$$

Hence, $\xi_S=\eta_S+m$, with $\eta\in core\left(\nu\right)$, $\eta\left(A\right)=\nu\left(A\right)$, and $m\left(G\right)\leq 0$ for all $G\subseteq A$. This proves Lemma 2 as long as $\xi\in L^1\left(P^*\right)$.

To extend the argument to any $\xi\in ca\left(\Sigma\right)$, we begin by observing that (15) implies

$$\xi_S\in L^*\partial\nu_e\left(1_{A_1}\right)+K_{A_1}^S$$

for all $A_1=\left(A\setminus N_1\right)\cup N_2\subseteq S$ with $P^*\left(N_1\right)=P^*\left(N_2\right)=0$. For, we have $\partial\nu_e\left(1_A\right)=\partial\nu_e\left(1_{A_1}\right)$, while the two cones K_A^S and $K_{A_1}^S$ agree over $L^1\left(P_S^*\right)$. Namely, $\left(K_A^S\bigtriangleup K_{A_1}^S\right)\cap L^1\left(P_S^*\right)=\{0\}$.

Now, let $\xi\in ca\left(\Sigma\right)$. By the Lebesgue Decomposition Theorem, we have $\xi=\xi^a+\xi^s$, where $\xi^a\in L^1\left(P^*\right)$ and $\xi^s\perp P^*$. Hence, there exists $N\in\Sigma$ such that $P^*\left(N\right)=0$ and $\left|\xi^s\right|\left(\Omega\setminus N\right)=0$. By the Hahn-Jordan Decomposition Theorem, there is a decomposition $N=N^+\cup N^-$ such that ξ^s is positive on N^+ and negative on N^-. By what has been proved before, there is $A\subseteq S$ such that

$$\xi_S^a\in L^*\partial\nu_e\left(1_A\right)+K_A^S.$$

Therefore,

$$\xi_S^a\in L^*\partial\nu_e\left(1_{A_1}\right)+K_{A_1}^S$$

with $A_1=\left(A\setminus N^+\right)\cup\left(S\cap N^-\right)$. On the other hand, it is easy to check that $\xi_S^s\in K_{A_1}^S$, and so

$$\xi_S=\xi_S^a+\xi_S^s\in L^*\partial\nu_e\left(1_{A_1}\right)+K_{A_1}^S,$$

which proves our claim for any $\xi\in ca\left(\Sigma\right)$. ■

Theorem 3. Having established Lemma 2, the proof of Theorem 3 is simple; the first part is similar to the argument used on p. 10 by [12].

As well-known, if V is any stable set of ν, then $core\left(\nu\right)\subseteq V$. It therefore suffices to prove that any element $\xi\in I\left(\nu\right)\setminus core\left(\nu\right)$ is dominated by some element in the core.

Suppose first that $\xi \in ca\,(\Sigma)$. As $\xi \in I\,(\nu)\setminus core\,(\nu)$, there is some $U \in \Sigma$ such that $\xi\,(U) < \nu\,(U)$. Let $\varepsilon > 0$ be such that $(\xi + \varepsilon P^*)\,(U) < \nu\,(U)$. Use Lemma 2 with $S = \Omega$ and with $\xi + \varepsilon P^*$ in place of ξ. Then, there exists $A \in \Sigma$ and $\eta \in core\,(\nu)$ such that $\eta\,(A) = \nu\,(A)$ and $\eta\,(B) \geq \xi\,(B) + \varepsilon P^*\,(B)$ for all $B \subseteq A$. Moreover, it holds $\nu\,(A) > \xi\,(A) + \varepsilon P^*\,(A)$ because $A \in \arg\max_{\Sigma} \nu - (\xi + \varepsilon P^*)$ and $\nu\,(U) - (\xi + \varepsilon P^*)\,(U) > 0$. Clearly, this implies $\eta\,(A) > \xi\,(A)$.

By Theorem 2, $\nu\,(B) = \min_{\chi \in R(\Gamma_\alpha)} \chi \cdot P\,(B)$. Hence, if $P^*\,(B) = 0$, the coalition B is null, and so we have $P^*\,(B) > 0$ for all non-null subcoalition B. Thus, $\eta\,(B) > \xi\,(B)$ for all non-null B. We conclude that $\eta \succ_A \xi$.

Next assume $\xi \in ba\,(\Sigma)$. The imputation ξ can be uniquely decomposed into a sum $\xi^c + \xi^p$, where $\xi^c \in ca\,(\Sigma)$ and ξ^p is purely additive. Therefore, there exists an increasing sequence $C_n \in \Sigma$ such that $|\xi^p|\,(C_n) = 0$ for all n, and $P^*\,(\Omega \setminus C_n) \to 0$ as $n \to \infty$ (see [27]). As before, let $U \in \Sigma$ be such that $\xi\,(U) < \nu\,(U)$. Consider the coalitions $U \cap C_n$. Since ν is continuous, there is some $\overline{n}$ such that $\xi\,(S) < \nu\,(S)$, with $S = U \cap C_{\overline{n}}$. For some $\varepsilon > 0$, we have $\xi\,(S) + \varepsilon P^*\,(S) < \nu\,(S)$ for some $\varepsilon > 0$. We now apply Lemma 2 where S is the same of the Lemma and ξ is replaced by $\xi^c + \varepsilon P^*$. Note that, by construction, $\xi\,(S) = \xi^c\,(S)$. There exists $A \subseteq S$ and $\eta \in core\,(\nu)$ such that $\eta\,(A) = \nu\,(A) > \xi^c\,(A) + \varepsilon P^*\,(A) = \xi\,(A) + \varepsilon P^*\,(A)$, and $\eta\,(B) \geq \xi\,(B) + \varepsilon P^*\,(B)$ for all $B \subseteq A$. The same argument used before implies that $\eta \succ_A \xi$ and the proof is completed. ■

Lemma 3. (i) Let $m_1, m_2 \in \overline{\Gamma}_\alpha$ and $E \in \Sigma$. There exist $m_1', m_2' \in \Gamma_\alpha$ such that $m_1\,(E) \geq m_1'\,(E)$ and $m_2\,(E) \geq m_2'\,(E)$. Hence, for every $t \in (0,1)$,

$$tm_1\,(E) + (1-t)\,m_2\,(E) \geq tm_1'\,(E) + (1-t)\,m_2'\,(E) \geq \min\{m_1'\,(E), m_2'\,(E)\}.$$

The set $\overline{\Gamma}_\alpha$ is therefore convex. It is also immediate to check that $\overline{\Gamma}_\alpha$ is closed. To show that it is weak*-compact, by the Alaoglu Theorem it is enough to prove that $\overline{\Gamma}_\alpha$ is norm bounded. Given $E \in \Sigma$, for each $m \in \overline{\Gamma}_\alpha$ there are $m', m'' \in \Gamma_\alpha$ such that $m'\,(E) \leq m\,(E) \leq m''\,(E)$ (consider E and E^c). Hence, $\sup_{m \in \overline{\Gamma}_\alpha} |m\,(E)| = \sup_{m \in \Gamma_\alpha} |m\,(E)| < \infty$ and $\overline{\Gamma}_\alpha$ is set-wise bounded. By the Uniform Boundedness Principle, $\overline{\Gamma}_\alpha$ is then norm bounded.

As to (ii), it is a straightforward consequence of (i). It remains to prove (iii). Set $\nu = \min_{m \in \overline{\Gamma}_\alpha} m$, which is well defined because $\overline{\Gamma}_\alpha$ is weak*-compact. To prove (iii) it is enough to show that $\overline{\Gamma}_\alpha = core\,(\nu)$. Clearly, $\overline{\Gamma}_\alpha \subseteq core\,(\nu)$. As to the converse, let $m \in core\,(\nu)$. By the definition of ν, for each E

there is $m' \in \overline{\Gamma}_\alpha$ such that $m'(E) \leq m(E)$. On the other hand, by the definition of m-closure, there is $m'' \in \Gamma_\alpha$ such that $m'(E) \geq m''(E)$. Hence, $m(E) \geq m''(E)$, and so $m \in \overline{\Gamma}_\alpha$. We conclude that $\overline{\Gamma}_\alpha = \mathit{core}(\nu)$, as desired. ■

Lemma 4. This proof is a variation on known arguments and we report it for the sake of completeness. Suppose first that ν_Γ is continuous at $\varnothing$ and at Ω. If we have $A_n \downarrow \varnothing$ and if $m \in \mathit{core}(\nu_\Gamma)$, we deduce that

$$\nu_\Gamma(A_n) \leq m(A_n) \leq \nu(\Omega) - \nu_\Gamma(\Omega \setminus A_n).$$

Hence, $m(A_n) \to 0$ provided ν_Γ is continuous at $\varnothing$ and Ω, and so $\overline{\Gamma} \subseteq \mathit{core}(\nu_\Gamma) \subseteq \mathit{ca}(\Sigma)$. Conversely, suppose $\Gamma \subseteq \mathit{ca}(\Sigma)$. Since Γ is weak*-compact, it is a weakly compact subset of $\mathit{ca}(\Sigma)$ (see, e.g., [29]). It is then easy to see that ν_Γ is continuous at $\varnothing$ and Ω.

(i) Let us show that $\overline{\Gamma} \subseteq \mathit{ca}(\Sigma)$. Assume $A_n \downarrow \varnothing$. Let us show that $m(A_n) \to 0$ for all $m \in \overline{\Gamma}$. By using the complementation, we have

$$m'_n(A_n) \geq m(A_n) \geq m''_n(A_n)$$

for all $m \in \overline{\Gamma}$, where m'_n and m''_n are elements in Γ. As Γ is weakly compact, the measures in Γ are uniformly additive (see [9], Th. 4.9.1). Hence, for all $\varepsilon > 0$, we have $|m'_n(A_m)| \leq \varepsilon$ for $m \geq m(\varepsilon)$ and for all n. In particular, $|m'_m(A_m)| \leq \varepsilon$. We conclude that $m'_n(A_n) \to 0$. The same argument leads to $m''_n(A_n) \to 0$, which, in turn, leads to $m(A_n) \to 0$. Hence, we have $m(A_n) \to 0$ for any sequence $A_n \downarrow \varnothing$, which means that $m \in \mathit{ca}(\Sigma)$. Hence, $\overline{\Gamma} \subseteq \mathit{ca}(\Sigma)$.

Next we prove that ν_Γ is continuous. Assume, by contradiction, that this is not the case. Then, there exists some $A \in \Sigma$ and a sequence, say $A_n \uparrow A$, and some $\eta > 0$ such that $|\nu(A_n) - \nu(A)| \geq \eta$. As ν is exact, we have $\nu(A_n) = m_n(A_n)$ with $m_n \in \overline{\Gamma}$. Since $\overline{\Gamma}$ is sequentially weakly compact, we can select a subsequence (renamed m_n) such that $m_n \to m^* \in \overline{\Gamma}$. We can then write

$$\nu(A_n) = m_n(A_n) = m_n(A) - m_n(A \setminus A_n)$$

As $n \to \infty$, we get $\lim_{n\to\infty} \nu(A_n) = m^*(A) \geq \nu(A)$, since $m_n(A \setminus A_n) \to 0$, thanks to the uniform additivity. On the other hand, there exists an $m \in \overline{\Gamma}$, such that $m(A) = \nu(A)$. Hence, from $m(A_n) \geq \nu(A_n)$, as $n \to \infty$, it follows $m(A) = \nu(A) \geq \lim_{n\to\infty} \nu(A_n) \geq \nu(A)$. We obtain $\lim_{n\to\infty} \nu(A_n) =$

$\nu(A)$. Therefore, the original sequence $\nu(A_n)$ has $\nu(A)$ as a limit point, contradicting $|\nu(A_n) - \nu(A)| \geq \eta$.

(ii) Since Γ is weakly compact, the existence of a positive measure λ such that $m \ll \lambda$ for all $m \in \Gamma_\alpha$ is guaranteed by Theorem IV.9.2 of [9]. As observed by [8], it is useful to modify slightly the construction in that theorem. In place of picking

$$\lambda = \sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{i=1}^{m_n} \frac{1}{2^i} |\mu_i^n|$$

with $\mu_i^n \in \Gamma$ in the proof of that theorem, we can use

$$\lambda = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{m_n} \sum_{i=1}^{m_n} |\mu_i^n| . \tag{16}$$

Clearly, we have $m \ll \lambda$ for all $m \in \overline{\Gamma}$. Actually, we have $m'(A) \geq m(A) \geq m''(A)$ for all A and for some $m', m'' \in \Gamma$. If $\lambda(A) = 0$, this implies $m'(A) = m''(A) = 0$. It follows that $m(A) = 0$, that means $m \ll \lambda$.

If $\Gamma \subseteq ca^+(\Sigma)$, in (16) we have $|\mu_i^n| = \mu_i^n$. As a consequence, we have

$$\lambda = \sum_{n=1}^{\infty} \frac{1}{2^n} \lambda_n$$

with $\lambda_n \in co(\Gamma)$. We infer that λ lies in the strong (and weak) closure of $co(\Gamma)$.

(iii) Assume $\Gamma \subseteq na(\Sigma) \cap ca(\Sigma)$. Obviously, the constructed positive measure λ of Eq. (16) is non-atomic. We must prove that any $m \in \overline{\Gamma} \subseteq L^1(\lambda)$ is non-atomic. Suppose, on the contrary, that A is an atom of m. Hence, $|m|(A) = \varepsilon > 0$. As $|m|$ is absolutely continuous w.r.t. λ, there is some $\delta > 0$ such that $\lambda(S) < \delta \implies |m|(S) < \varepsilon/2$. On the other hand, as λ is non-atomic, we can find a subcoalition $A_1 \subseteq A$, for which $\lambda(A_1) < \delta$. Hence, $|m|(A_1) < \varepsilon/2$ which is a contradiction. ■

Lemma 5. (i) Observe that $\Delta = \Gamma \ominus G$ amounts to $\Delta = \cap_{g\in G}(\Gamma - g)$. Hence, Δ turns out to be the intersection of a family of the weak*-compact sets $\Gamma - g$. (ii) Let $\Delta = \Gamma \ominus G$. By definition, we have $G + \Delta \subseteq \Gamma$. It is immediately checked that $G + \overline{\Delta} \subseteq \Gamma$, provided Γ is m-closed. By the maximality of the set Δ, we infer $\Delta = \overline{\Delta}$. (iii) Suppose $\overline{\Gamma} \ominus G \neq \varnothing$ and denote $\Delta = \overline{\Gamma} \ominus G$. This implies $G + \Delta \subseteq \overline{\Gamma}$. Hence, $\nu_G + \nu_\Delta \geq \nu_{\overline{\Gamma}} = \nu_\Gamma$. We deduce that

$$\nu_{\overline{\Gamma} \ominus G} \geq \nu_\Gamma - \nu_G. \tag{17}$$

Since $\nu_{\overline{\Gamma} \ominus G}$ is exact, the core of $\nu_\Gamma - \nu_G$ is non-empty. Conversely, suppose $core\,(\nu_\Gamma - \nu_G) \neq \varnothing$. Let μ be any exact game such that $\mu \geq \nu_\Gamma - \nu_G$ and $\mu\,(\Omega) = \nu_\Gamma\,(\Omega) - \nu_G\,(\Omega)$. Clearly, some μ exists, for example the exact envelope $(\nu_\Gamma - \nu_G)_e$ is of this kind. Now, fix some $A \in \Sigma$ and take two any elements $m \in core\,(\mu)$ and $g \in G$. The inequality $\mu \geq \nu_\Gamma - \nu_G$ implies that $m\,(A) + g\,(A) \geq \nu_\Gamma\,(A)$. Consequently, we have $m + g \in \overline{\Gamma}$. Namely, $core\,(\mu) + G \subseteq \overline{\Gamma}$, that, in turn, implies $core\,(\mu) \subseteq \overline{\Gamma} \ominus G$. As first consequence, $\overline{\Gamma} \ominus G \neq \varnothing$ and this proves the converse implication. Moreover, we have obtained that $\mu \geq \nu_{\overline{\Gamma} \ominus G}$. In view of (17), $\nu_{\overline{\Gamma} \ominus G}$ is the minimal exact game greater than $\nu_\Gamma - \nu_G$. Therefore, $\nu_{\overline{\Gamma} \ominus G} = (\nu_\Gamma - \nu_G)_e$. On the other hand, we have

$$\overline{\Gamma} \ominus G = core\left(\nu_{\overline{\Gamma} \ominus G}\right) = core\,(\nu_\Gamma - \nu_G)_e = core\,(\nu_\Gamma - \nu_G)$$

where the first equality comes from (i). ■

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