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Working Paper no. 14/2003
June 2003

APPLIED MATHEMATICS
WORKING PAPER SERIES

Choquet Insurance Pricing: a Caveat*

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Submitted: December 2002
Revised: May 2003

1 Introduction

Some recent papers have used Choquet expectations with respect to nonadditive probabilities as pricing functionals for insurance and financial markets in order to capture the presence of frictions or the uncertainty aversion of the agents. See, for example, Wang [7] and Wang, Young, and Panjer [8] for insurance markets, Chateauneuf, Kast, and Lapied [2] for financial markets, and Wang [9] for both.

In this note we show that this class of price functionals represents "strong" frictionalities, as the presence on the market of assets without bid-ask spreads may turn such prices into standard expectations, thus making the whole market frictionless.

Specifically, in the setup of [7], [8], and [9] – where the price functional depends only on the distribution of the asset under a given probability measure – our Theorem 1(i) shows that the existence of *any* risky frictionless asset makes the whole market frictionless. Moreover, in Theorem 1(ii) we show that in the "distribution-free" setup of [2], the same collapse occurs provided there exists a frictionless and fully revealing asset. Clearly, the latter assumption is

*We wish to thank Anna Battauz, two anonymous referees, and an associate editor for very helpful suggestions. The financial support of MIUR and NOMOS is gratefully acknowledged.

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less likely to occur in applications than the former; it is, however, of theoretical interest as a benchmark case.

Though the focus of our note is on pricing, similar remarks apply to the risk measures of Artzner, Delbaen, Eber and Heath [1] and Delbaen [4] when they depend only on the distribution of the risky position, as it is the case for the worst conditional expectation (section 5 of [1] and remark 7.7 of [4]) and, more generally, for any distorted risk measure (section 4 of [4]).

2 Setup

We consider a potentially incomplete market and one period of uncertainty $(0, T)$. The set of states of nature at time T is represented by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and a *financial asset* (or an insurance contract) is a random variable $X : \Omega \to \mathbb{R}$. Let $\mathcal{X}$ be the vector space of all the marketed assets. For convenience, we will only consider the case in which $\mathcal{X}$ consists of essentially bounded random variables; that is, $\mathcal{X} \subseteq L^\infty = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$. We denote by L_0^∞ the subset of L^∞ consisting of all simple random variables.

An asset M is *risky* if it is not almost surely constant. A *cash-or-nothing call* M on a stock S is an option that pays nothing if the underlying stock's price ends up below the strike price k at time T, and pays a fixed amount a if it ends up above the strike price; that is, $M = a1_{\{S>k\}}$. The knowledge of $M(\omega)$ reveals whether the event $\{S > k\}$ obtained or not. An asset M is *fully revealing* if the σ-algebra generated by M coincides with $\mathcal{F}$; that is, all available information is summarized by M. For example, by a classic result of Mackey [5] all injective random variables are fully revealing on a standard Borel space.[1]

A *price* on $\mathcal{X}$ is just a functional $\pi : \mathcal{X} \to \mathbb{R}$. We say that it *only depends on the distribution under* $\mathbb{P}$ if any two assets sharing the same distribution with respect to $\mathbb{P}$ have the same price. The market for an asset M is frictionless if there is no bid-ask spread for M at price π, that is, $\pi(-M) = -\pi(M)$. With a little abuse, in this case we will say that M is *frictionless.*

A *(submodular) nonadditive probability* is a set function $\mathbb{W} : \mathcal{F} \to [0, 1]$ such that:

(i) $\mathbb{W}(\emptyset) = 0$ and $\mathbb{W}(\Omega) = 1$,

(ii) $\mathbb{W}(A) \leq \mathbb{W}(B)$ if $A \subseteq B$,

(iii) $\mathbb{W}(A_n) \downarrow 0$ if $A_n \downarrow \emptyset$,

(iv) $\mathbb{W}(A \cup B) \leq \mathbb{W}(A) + \mathbb{W}(B) - \mathbb{W}(A \cap B)$ for all A, B.

[1] $(\Omega, \mathcal{F})$ is a *standard Borel space* if it is isomorphic to a pair $(\Omega', \mathcal{F}')$, where Ω' is a Borel subset of some Polish space and $\mathcal{F}'$ is its Borel σ-algebra.

Standard countably additive probabilities are the special case in which we have an equality in (iv). We call them *probability measures.*

A nonadditive probability $\mathbb{W}$ is *absolutely continuous with respect to* $\mathbb{P}$, written $\mathbb{W} \ll \mathbb{P}$, if $\mathbb{P}(A) = 0$ implies $\mathbb{W}(A) = 0$. In this case, for all $X \in L^\infty$ the *Choquet expectation of* X *with respect to* $\mathbb{W}$ is given by

$$\mathbf{E}_{\mathbb{W}}[X] = \int_{-\infty}^{0} (\mathbb{W}\{X > t\} - 1)\, dt + \int_{0}^{\infty} \mathbb{W}\{X > t\}\, dt. \tag{1}$$

When $\mathbb{W}$ is probability measure, then (1) reduces to the standard expectation. It can be shown that

$$\mathbf{E}_{\mathbb{W}}[X] = \sup\{\mathbf{E}_{\mathbb{Q}}[X] : \mathbb{Q} \in \mathfrak{P}\}, \tag{2}$$

where $\mathfrak{P}$ is the set of all probability measures $\mathbb{Q}$ such that $\mathbb{Q}(A) \leq \mathbb{W}(A)$ for all $A \in \mathcal{F}$.

3 Caveat

We begin with a lemma of independent interest that applies to price functionals that are suprema of expectations. By (2), Choquet expectations belong to this class.

Lemma 1 *Let* $\mathfrak{P}$ *be a family of probability measures that are absolutely continuous with respect to a nonatomic probability measure* $\mathbb{P}$*, and let* $\pi(X) = \sup\{\mathbf{E}_{\mathbb{Q}}[X] : \mathbb{Q} \in \mathfrak{P}\}$ *for all* $X \in \mathcal{X}$*. If* $\mathcal{X} \supseteq L_0^\infty$ *and* π *depends only on the distribution under* $\mathbb{P}$*, then* $\mathbb{Q} = \mathbb{P}$ *for all* $\mathbb{Q} \in \mathfrak{P}$ *provided there exists a risky frictionless cash-or-nothing call.*

In the Choquet case we have a stronger result.

Theorem 1 *Let* $\mathbb{W} \ll \mathbb{P}$ *be a nonadditive probability, and let* $\pi(X) = \mathbf{E}_{\mathbb{W}}[X]$ *for all* $X \in \mathcal{X}$*. If there exists a risky frictionless asset* M *in* X*, then* $\mathbb{W}$ *is a probability measure provided at least one of the following two conditions holds:*

(i) $\mathcal{X} \supseteq L_0^\infty$*,* $\mathbb{P}$ *is nonatomic, and* π *depends only on the distribution under* $\mathbb{P}$*. In this case* $\mathbb{W} = \mathbb{P}$*.*

(ii) M *is fully revealing.*

As suggested in the introduction, our results say that if prices are Choquet expectations, then the existence of *one* frictionless asset may force the whole market to be frictionless. Any risky asset will cause this collapse if $\mathbb{P}$ is nonatomic and π depends only on the distribution, while the frictionless asset has to be fully revealing if such dependence is not required.

More precisely, in the setup of [7], [8], and [9], point (i) means that if an insurance firm calculates risk loads according to a probability distortion and second order stochastic dominance, then it assigns either strictly positive risk load to all contracts or zero load to all contracts. In the setup of [2], by point (ii) the existence of a state revealing frictionless asset is necessary and sufficient to guarantee that the whole market is frictionless.

A Proofs

Lemma 1. W.l.o.g., set $M = 1_A$ for some $A \in \mathcal{F}$ such that $0 < \mathbb{P}(A) < 1$. Since $\pi(1_A) = -\pi(-1_A)$, then $\sup_{\mathfrak{P}} \mathbb{Q}(A) = -\sup_{\mathfrak{P}} -\mathbb{Q}(A) = \inf_{\mathfrak{P}} \mathbb{Q}(A)$. Therefore $\mathbb{Q}(A) = \mathbb{Q}'(A) = \pi(1_A)$ for all $\mathbb{Q}, \mathbb{Q}' \in \mathfrak{P}$. Let $\alpha = \pi(1_A)$. Let $B \in \mathcal{F}$ satisfy $\mathbb{P}(B) = \mathbb{P}(A)$, then $F_{1_B} = F_{1_A}$ and $F_{-1_B} = F_{-1_A}$;[2] but π depends only on the distribution of the random variable whence $\pi(1_B) = \pi(1_A) = -\pi(-1_A) = -\pi(-1_B)$. Then, by the argument just used, $\mathbb{Q}(B) = \mathbb{Q}'(B) = \alpha = \mathbb{Q}(A)$ for all $\mathbb{Q}, \mathbb{Q}' \in \mathfrak{P}$. That is: $\mathbb{P}(B) = \mathbb{P}(A)$ implies $\mathbb{Q}(B) = \mathbb{Q}(A)$ for all $\mathbb{Q} \in \mathfrak{P}$. By [6, Thm 1], $\mathbb{Q} = \mathbb{P}$ for all $\mathbb{Q} \in \mathfrak{P}$, which concludes the proof. ■

Theorem 1. Since M is almost surely bounded, w.l.o.g. we can assume $M \geq 0$ almost surely (hence $M \geq 0$ $\mathbb{Q}$-almost surely for all $\mathbb{Q} \in \mathfrak{P}$). Then,

$$\sup_{\mathbb{Q}' \in \mathfrak{P}} \mathbf{E}_{\mathbb{Q}'}[M] = \pi(M) = -\pi(-M) = -\sup_{\mathbb{Q}' \in \mathfrak{P}} \mathbf{E}_{\mathbb{Q}'}[-M] = \inf_{\mathbb{Q}' \in \mathfrak{P}} \mathbf{E}_{\mathbb{Q}'}[M],$$

that is,

$$\mathbf{E}_{\mathbb{Q}}[M] = \pi(M) \tag{3}$$

for all $\mathbb{Q} \in \mathfrak{P}$. Notice that, if $t < 0$ and $\mathbb{Q} \in \mathfrak{P}$,

$$\mathbb{W}(\{M > t\}) = \sup_{\mathbb{Q}' \in \mathfrak{P}} \mathbb{Q}'(\{M > t\}) = 1 = \mathbb{Q}(\{M > t\}).$$

Therefore, (3) yields

$$\int_0^{\|M\|_\infty} \mathbb{Q}(\{M > t\})\, dt = \int_0^{\|M\|_\infty} \mathbb{W}(\{M > t\})\, dt \tag{4}$$

since $t \geq \|M\|_\infty$ implies $\mathbb{W}(\{M > t\}) = 0 = \mathbb{Q}(\{M > t\})$ for all $\mathbb{Q} \in \mathfrak{P}$.
It is easy to prove that the function $t \mapsto \mathbb{W}(\{M > t\})$ is (weakly) decreasing and right continuous, and so it is $t \mapsto \mathbb{Q}(\{M > t\})$ (it is just the decreasing distribution function of M w.r.t. $\mathbb{Q}$). Moreover, $\mathbb{W}(\{M > t\}) \geq \mathbb{Q}(\{M > t\}))$ for all $t \in \mathbb{R}$. Then (4) yields

$$\mathbb{Q}(\{M > t\}) = \mathbb{W}(\{M > t\}) \tag{5}$$

for all $t \in [0, \|M\|_\infty)$. Since we already observed that (5) holds also if $t \notin [0, \|M\|_\infty)$, we conclude that

$$\mathbb{Q}(\{M > t\}) = \mathbb{W}(\{M > t\}) \quad \forall t \in \mathbb{R} \;\; \forall \mathbb{Q} \in \mathfrak{P}. \tag{6}$$

[2] F_X denotes the distribution function of X w.r.t. $\mathbb{P}$.

In case (ii), this implies that all the measures in $\mathfrak{P}$ coincide on the π-class $\{\{M > t\} : t \in \mathbb{R}\}$, which generates $\mathcal{F}$ since M is state revealing. We can conclude that they coincide on $\mathcal{F}$.
In case (i), for all $t \in \mathbb{R}$,

$$\pi\left(1_{\{M>t\}}\right) = \sup_{\mathbb{Q}\in\mathfrak{P}} \mathbb{Q}\left(\{M > t\}\right) = -\sup_{\mathbb{Q}\in\mathfrak{P}} -\mathbb{Q}\left(\{M > t\}\right) = -\pi\left(-1_{\{M>t\}}\right).$$

Since M is not a.s. constant, there exists $\bar{t} \in \mathbb{R}$ such that $0 < \mathbb{P}\left(\{M > \bar{t}\}\right) < 1$, that is $1_{\{-M>\bar{t}\}}$ is not a.s. constant. Apply Lemma 1. ■

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