ASYMPTOTIC EFFICIENCY OF SIGNED - RANK SYMMETRY TESTS UNDER SKEW ALTERNATIVES

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Abstract

The efficiency of some known tests for symmetry such as the sign test, the Wilcoxon signed-rank test or more general linear signed rank tests was studied mainly under the classical alternatives of location. However it is interesting to compare the efficiencies of these tests under asymmetric alternatives like the so-called skew alternative proposed in Azzalini (1985). We find and compare local Bahadur efficiencies of linear signed-rank statistics for skew alternatives and discuss also the conditions of their local optimality.We calculate also such efficiencies for the family of distribution-free Maesono statistics proposed in Maesono (1987).

Key words : skew family, linear rank test, Maesono statistic, Bahadur efficiency, local optimality.

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1 Introduction

One of most interesting and important problems of classical Nonparametric Statistics is the testing of symmetry. Numerous signed-rank distribution-free tests were proposed to that end (see, e.g., Hájek and Sidak (1967) or Hettmansperger (1984) for their description). Their properties such as limiting distributions, rate of convergence, consistency, etc. have been well-studied. The asymptotic comparison of such tests by their efficiency was carried out on the basis of Pitman, Bahadur, Chernoff and Hodges-Lehmann efficiencies(see, e.g., Chernoff and Savage (1958), Hettmansperger (1984) and Nikitin (1995)). It is interesting to note that all these types of efficiency give essentially the same ordering of linear signed-rank tests but under slightly different regularity conditions.

Most examples of efficiency calculations include classical alternative of location (see, e.g.,Groeneboom (1981), Kremer (1982), Nikitin (1995)). However, in many practical problems this model retaining the symmetry and the form of the hypothetical distribution is unrealistic as the alternative distributions are often skewed and lose their symmetry. An interesting asymmetric alternative model in the normal case was introduced in Azzalini (1985) .

Let Φ and φ denote the distribution function (d.f.) and the density of the standard normal law. Azzalini Azzalini (1985), Azzalini (1986) proposed the skew-normal distribution depending on the real parameter θ and having the density

$$p(x,\theta) = 2\varphi(x)\Phi(\theta x),\ x \in R^1,\ \theta \geq 0.$$

Clearly for any θ the function $p(x,\theta)$ is a density and for $\theta = 0$ we obtain the standard normal density. Later the properties of Azzalini model and its generalizations were considered in Henze (1986), Salvan (1986),Johnson et al. (1988), Liseo (1990), Azzalini and Dalla Valle (1996), and Chiogna (1998).

As mentioned in Azzalini (1985), for any symmetric distribution function F with density f we can introduce analogously the skew distribution with the density

$$h(x,\theta) = 2f(x)F(\theta x),\ x \in R^1, \theta \geq 0. \tag{1}$$

The interest in skew models considerably increased in the last years which can be seen in recent papers of Mateu-Figuera (1998), Azzalini and Capitanio (1999), Arnold and Beaver (2000), Pewsey (2000), Genton et al. (2001), among others.

Therefore it is quite interesting to find out the efficiencies of signed-rank tests mentioned above with respect to skew alternatives. General formulas for local Bahadur efficiencies for the linear signed-rank tests in the case of regular one-parametric families can be found in (Nikitin , 1995, Chap. 4). However the alternative (1)requires some study of its structure partially done in Durio and Nikitin (2001a), Durio and Nikitin (2001b). Section 2 is devoted to the formulation of the regularity conditions and to the description of considered tests.

We give special attention to the signed rank tests with the "power" score function. Besides the signed-rank tests we study also the family of symmetry tests proposed in Maesono (1987). They can be considered as the generalizations of Wilcoxon test.

In Section 3 we give general formulas for the local Bahadur efficiency of considered tests and calculate this efficiency for five different model distributions : normal, logistic, arcsine, uniform and for the distribution with the density

$$f_5(x) = (8/3\pi)(1+x^2)^{-3}, \ x \in R^1 \tag{2}$$

which has power tails and resembles the Cauchy distribution.

The results of Section 3 demonstrate that the ordering of tests by their efficiency is more or less similar to the location case but the exact values of efficiency coincide only for the normal law.

In Section 4 we discuss the conditions of local Bahadur optimality of our tests under skew alternatives. We show that some of our tests are never locally optimal in a broad class of symmetric probability laws. For other tests this form of optimality is possible though for entirely different parent distributions in comparison to the location case, see Chapter 6 of Nikitin (1995).

In the sequel we denote by $C_1, C_2, ...$ different positive constants.

2 Regularity Conditions and Tests

Let $X_1, ..., X_n$ be a sample with the unknown density $g(x)$. We want to test the symmetry hypothesis

$$H_0 : g(x) = g(-x), \ x \in R^1. \tag{3}$$

The alternative H_1 consists in that the observations have the density (1) where the density f is known and is supposed to be symmetric. Denote by $H(x,\theta)$ the d.f. corresponding to this density. We require that the density f has a finite variance, that $f(0) > 0$ and that f is positive and differentiable within its support; by symmetry we always have $f'(0) = 0$.

Moreover we assume that the density f satisfies uniformly in $x \in R^1$ the following condition.

Condition 1:

$$H(x,\theta) - F(x) \sim 2\theta f(0) \int_{-\infty}^{x} uf(u)du, \quad as \quad \theta \to 0. \tag{4}$$

We can easily formulate sufficient conditions ensuring (4). For instance assuming the existence of bounded f' we can write for any x

$$H(x,\theta) - F(x) = 2\theta f(0) \int_{-\infty}^{x} uf(u)du + \theta^2 \int_{-\infty}^{x} u^2 f(u) f'(\xi u)du, \, 0 < \xi < x.$$

The last term is of order $O(\theta^2)$, its contribution is negligible and we get (4).

An important quantity when testing of symmetry is the corresponding variant of Kullback - Leibler information

$$K_h(\theta) := \int_{R^1} \ln \frac{2h(x,\theta)}{h(x,\theta)+h(-x,\theta)} h(x,\theta)dx,$$

see its definition and properties in Section 4.4 of Nikitin (1995). In our notations due to the symmetry of f we have

$$K_h(\theta) = K(f,\theta) := 2\int_{R^1} \ln\{2F(\theta x)\} f(x)F(\theta x)dx. \tag{5}$$

Another condition we impose on f is the following local behavior of $K(f,\theta)$ as $\theta \to 0$:
Condition 2:

$$K(f,\theta) \sim 2f^2(0)\int_{R^1} x^2 f(x)dx \cdot \theta^2 \,. \tag{6}$$

In fact, this behavior takes place under the assumption that the second derivative of f is bounded in the neighborhood of zero and that there exists the absolute moment of third order. The detailed proof of this assertion can be found in Durio and Nikitin (2001a).

Note that the first term in the asymptotics of $K(f,\theta)$ coincides with the Fisher information $I(f)$ in the point $\theta = 0$ that is in our case

$$I(f) = 2\int_{R^1} \frac{(xf(x\theta))^2}{F(x\theta)} f(x)dx|_{\theta=0}.$$

Under further regularity conditions it is possible to prove that

$$I(f) = 4f^2(0)\int_{R^1} x^2 f(x)dx = 4f^2(0)VarX_1.$$

However we prefer not including such conditions and will simply use the asymptotics (6) obtained for $K(f,\theta)$.

It is important to observe that the conditions (4) and (6) are sometimes valid even without the condition of boundedness for the density's derivative. An interesting example is the symmetric arcsine density

$$f(x) = (\pi\sqrt{1-x^2})^{-1}\,\mathbf{1}\{-1 \le x \le 1\}. \tag{7}$$

This density and its derivative are clearly unbounded on $[-1,1]$. However (4) and (6) are true as proved in Durio and Nikitin (2001a).

Denote the set of symmetric densities f satisfying all requirements stated above including conditions 1 and 2 by $\mathcal{F}$.

Now we pass to the description of tests. The simplest distribution-free test for testing of symmetry is certainly the sign test based on the statistic

$$S_n = n^{-1} \sum_{i=1}^{n} 1\{X_i > 0\}. \tag{8}$$

We are interested also in the signed-rank tests based on more complicated statistics of the form

$$Z_n = n^{-1} \sum_{i=1}^{n} a_n \left(\frac{R_i^+}{n+1} \right) 1\{X_i > 0\}, \tag{9}$$

where $(R_1^+, ..., R_n^+)$ is the vector of absolute ranks for the sample $|X_1|, ..., |X_n|$ and the function a_n is supposed to be defined on [0,1] and to be constant on the intervals of the form $(\frac{i-1}{n}, \frac{i}{n}], 1 \leq i \leq n$. We also assume that one has convergence in quadratic mean

$$a_n(u) \to J(u), \ n \to \infty,$$

where $J(u)$ is some function on $[0, 1]$ called the score function.

The sign test is a particular case of statistics Z_n in case of constant score function, but we prefer to study it separately.

When studying statistics (9) we consider only score functions J with bounded derivative on [0, 1] and such that $J(0) = 0$. To simplify the formulas for the efficiency we assume that J is normalized in such a way that

$$\int_0^1 J^2(u)du = 4. \tag{10}$$

Typical example of such score functions is the linear function $J_1(u) = \sqrt{12}\, u$ which corresponds to the Wilcoxon signed-rank test. We may consider also the "power" score function $J_r(u) = 2\sqrt{2r+1}\, u^r, r > 1$. It is also interesting to try the "trigonometric" score function $J^*(u) = \sqrt{8} \sin(\pi u/2)$.

Another family of statistics for testing of symmetry which we denote by $M_r, r = 2, 3...$, was proposed by Maesono (1987). These statistics are the generalizations of Wilcoxon statistic W coinciding with it for $r = 2$. Maesono showed that the statistics M_r, $r > 2$ are sometimes more efficient in Pitman and Bahadur senses than the Wilcoxon statistic $W = M_2$ and hence deserve to be used. Some further properties of Maesono statistics were studied in Nikitin and Ponikarov (2001).

The Maesono statistic of any natural order $r \geq 2$ is a U-statistic

$$M_r = \binom{n}{r}^{-1} \sum_{1 \leq i_1 < ... < i_r \leq n} K_r(X_{i_1}, ..., X_{i_r})$$

with the kernel

$$K_r(s_1, ..., s_r) = r^{-1} \left(\sum_{i=1}^{r} \prod_{j \neq i}^{r} \mathbf{1}_{(s_i + s_j > 0)} - 1 \right).$$

It is evident that these statistics are distribution-free under H_0.

For a clear idea of kernels K_r we write out such kernel for $r = 3$

$$K_3(s_1, s_2, s_3) = \frac{1}{3}(\mathbf{1}_{(s_1+s_2>0, s_1+s_3>0)} + \mathbf{1}_{(s_1+s_2>0, s_2+s_3>0)} + \mathbf{1}_{(s_2+s_3>0, s_1+s_3>0)} - 1).$$

It was proved in Nikitin and Ponikarov (2001) that the statistics M_r under H_0 are asymptotically normal with zero mean and variance

$$\sigma_r^2 = 2r^{-2}(\frac{1}{2r-1} - \frac{(r-1)!^2}{(2r-1)!}).$$

This result simplifies their use for testing of symmetry in large samples.

3 Calculation of Local Efficiencies

The efficiency of the sign test and of signed-rank tests was studied by several authors on the basis of Pitman, Bahadur, Chernoff and Hodges-Lehmann efficiency, see the references in Hettmansperger (1984), Nikitin (1995). However the calculations were done mainly for location alternatives. We find below the values of efficiency for skew alternatives. General formulas for the efficiencies of linear signed rank tests in case of sufficiently regular parametric alternatives can be found in Nikitin (1995). It is known that the formulas for the calculation of efficiency are essentially the same for all four types of efficiency, the difference is only in regularity conditions imposed on J and the family (1). Hence we will make the calculations for Bahadur efficiency where the regularity conditions are less stringent and their verification is easier.

This type of efficiency was introduced and developed in Bahadur (1967),Bahadur (1971). The measure of local Bahadur efficiency for a sequence of statistics $\{T_n\}$ used to test H_0 against H_1 is the local exact slope $c_T(f, \theta)$ which is calculated according to certain rule taking in account the large deviation behavior of the statistic under the null-hypothesis and the limit in probabilty or almost surely under the alternative. General formulas for the local slopes of signed-rank statistics are given in Nikitin (1995), p.144. Namely

$$c_S(f, \theta) = 4[H(0, \theta) - 1/2]^2,$$

and

$$c_Z(f, \theta) = [\int_0^{+\infty} J(H(x, \theta) - H(-x, \theta))dH(x, \theta) - J_0]^2,$$

where $J_0 = \frac{1}{2}\int_0^1 J(u)du$.

Consider now the so-called local indices $l(T, f)$ which are defined as follows:

$$l(T, f) := \lim_{\theta \to 0} c_T(f, \theta)/\theta^2 \qquad (11)$$

Using (4) we can obtain the following explicit formulas for local indices of statistics S_n and Z_n. For the sign test we have

$$l(S,f) = 16f^2(0)\{\int_{-\infty}^{0} xf(x)dx\}^2 = 4f^2(0)E|X_1|^2. \tag{12}$$

For statistics (9) we use the boundedness of J' under which we get easily

$$l(Z,f) = 4f^2(0)\{\int_{0}^{\infty}(J(2F(x)-1)xf(x)dx\}^2. \tag{13}$$

This formula shows that for small θ one has the inequality $c_Z(f,\theta) > 0$ which guarantees the consistency of the test based on Z_n.

The Bahadur efficiency of statistics M_r was studied only for the location alternative in Nikitin and Ponikarov (2001). It follows easily from the Law of Large Numbers for U-statistics, see Korolyuk et al. (1994) that under the alternative 1 we have convergence in probability

$$M_r \to E_\theta(M_r) := b_r(\theta) = \int_{R^1}(1-H(-x,\theta))^{r-1}dH(x,\theta) - \frac{1}{r}.$$

Under mild conditions imposed on f which are satisfied for five densities introduced above we have as $\theta \to 0$:

$$b_r(\theta) \sim 4\theta f(0)\int_{R^1} F^{r-1}(x)xf(x)dx.$$

Taking in account the large deviation asymptotics of M_r which was found in Nikitin and Ponikarov (2001) we get the following formula for the local exact slope of the statistic M_r:

$$c_{M_r}(\theta) \sim b_r^2(\theta)/r^2\sigma_r^2 \sim 8\theta^2 f^2(0)(\int_{R^1} F^{r-1}(x)xf(x)dx)^2/r^2\sigma_r^2. \tag{14}$$

Hence the local index of Maesono statistic M_r is

$$l(M_r,f) = 8f^2(0)(\int_{R^1} F^{r-1}(x)xf(x)dx)^2/r^2\sigma_r^2. \tag{15}$$

It is easy to verify that for $r=2$ this value coincides with the local index of Wilcoxon statistic given above.

It is curious that for $r=3$ this value is the same. The proof is as follows. Using the symmetry of F we get easily

$$\int_{R^1} F^{r-1}(x)xf(x)dx = -\int_{R^1}(1-F(x))^{r-1}xf(x)dx$$

and hence the local index of M_r from (15) can be written in the form

$$l(M_r,f) = 4f^2(0)(\int_{R^1}[F^{r-1}(x)-(1-F(x))^{r-1}]xf(x)dx)^2/(r^2\sigma_r^2). \tag{16}$$

Now it is evident from (16) that $l(M_2, f) = l(M_3, f)$ that corresponds to the similar observation in Maesono (1987) in for the location alternative. However the tests based on M_2 and M_3 are not identical that can be seen from their nonlocal behavior, see Nikitin and Ponikarov (2001) for details.

We will calculate the local indices for five symmetric densities f defined on R^1. The first is the standard normal density $f_1(x) = (2\pi)^{-1/2}\exp(-x^2/2)$, the second is the logistic one $f_2(x) = \exp(x)/(1+\exp(x))^2$, the third is the arcsine density $f_3(x) = (\pi\sqrt{1-x^2})^{-1}\mathbf{1}\{-1 \le x \le 1\}$, the fourth density f_4 is the uniform one on [-1, 1] and the fifth density f_5 is given by (2).

It is easy to see that for these densities the quantity $E|X_1|$ required for the efficiency of the sign test is respectively $\sqrt{2/\pi}, 2\ln 2, 2/\pi, 1/2$ and $4/(3\pi)$.

From general theory it is known (Nikitin , 1995, Chap. 4) that for any sequence of statistics T_n and all θ one has the so-called Bahadur - Raghavachari inequality

$$c_T(f,\theta) \le 2K(f,\theta)$$

so that the local (absolute) Bahadur efficiency of the sequence T_n is defined as

$$e^B(T,f) := \lim_{\theta\to 0} c_T(f,\theta)/2K(f,\theta).$$

Hence according to (6) and (12) we get

$$e^B(S,f) = (E|X_1|)^2/VarX_1,$$

and using (6) and (13)

$$e^B(Z,f) = \{\int_0^\infty (J(2F(x)-1)xf(x)dx\}^2/VarX_1.$$

For Maesono statistics we get analogously

$$e^B(M_r,f) = 4(\int_{R^1} F^{r-1}(x)xf(x)dx)^2/(r^2\sigma_r^2 \cdot VarX_1).$$

Note that the variance $VarX_1$ is in our five cases respectively 1, $\pi^2/3$, 1/2, 1/3 and 1/3. The results of calculations for our five densities and four statistics for testing of symmetry : sign S, Wilcoxon W, Maesono $M_4 - M_6$, "power" of order 2,3 and 4 "trigonometric" Tr are given in the Table 1.

Now let us discuss some issues from this table.

First, we see that the efficiency of our tests is rather high with the possible exception of the sign test. It means that our tests discriminate well the large class of asymmetric skew distributions from symmetric ones. Second, comparing the column for the normal law with (Nikitin , 1995, Chap. 2) and Nikitin and Ponikarov (2001) we recognize for the sign, Wilcoxon and Maesono statistic of order 4 the same values of local indices as in the location case. This invariance is a consequence of the equation

$$f_1(x) = -xf_1(x)$$

Statistic	Distribution				
	Gauss	Logistic	Arcsine	Uniform	f_5
S	0.637	0.584	0.812	0.750	0.540
W	0.955	0.912	0.986	1	0.862
M_4	0.976	0.941	0.963	0.995	0.895
M_5	0.988	0.964	0.922	0.974	0.923
M_6	0.985	0.972	0.869	0.939	0.938
P_2	0.977	0.961	0.866	0.938	0.925
P_3	0.939	0.946	0.743	0.840	0.923
P_4	0.887	0.912	0.641	0.750	0.901
Tr	0.900	0.845	1	0.986	0.791

Table 1: Local Bahadur efficiencies under skew alternatives.

which is a characteristic property of the symmetric normal law and is not preserved for other distributions. Hence the curious phenomenon of the same local Bahadur efficiencies for the skewed and shifted distribution for the signed-rank tests for symmetry is in fact a characterization of the normal law in a very broad class of probability laws. This property was already discovered for goodness-of-fit tests in Durio and Nikitin (2001a),Durio and Nikitin (2001b).

Third, the comparison with the results of (Nikitin , 1995, Section 4.5) shows that the ordering of tests is similar to the location case. This is favourable for practitioners: as they seldom know the structure of the alternative they can make the same choice of test both for location and skew alternative models.

Fourth, we must underline local optimality of W for the uniform distribution (it means local optimality of the Wilcoxon test) and the local optimality of the signed-rank test with "trigonometric" score for the arcsine distribution. In the next section we give the theoretical analysis of this observation.

Fifth, we observe that the efficiency of linear signed-rank tests with the power score function depend on the power and the maximum of this efficiency is attained on different powers for different distributions. Similar situation takes place for Maesono statistics. To illustrate this we give below two figures showing how the efficiency changes as a function of power r and of parent distribution.

On the figure 1 we give the efficiencies of linear signed-rank statistics with different power scores. It is seen that for uniform and arcsine distributions the plots are monotonic that signifies that the maximum of efficiency is attained for minimal power. However for three remaining distributions the maximal efficiency is attained somewhere for small but not minimal values of power and after that

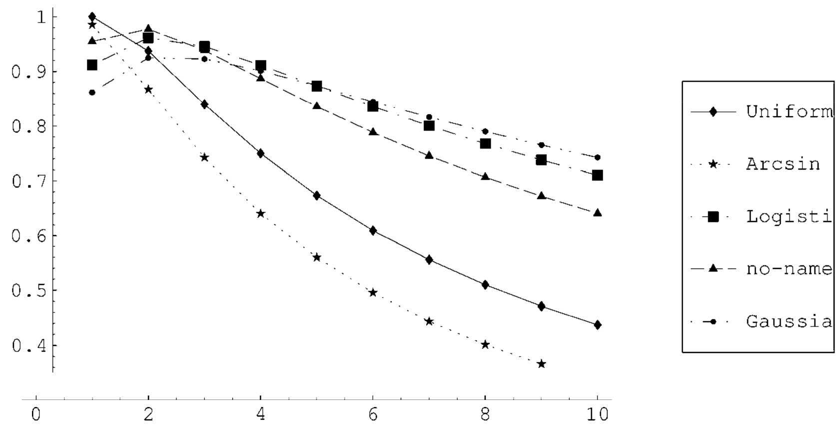

Figure 1: Efficiences of tests for different power score

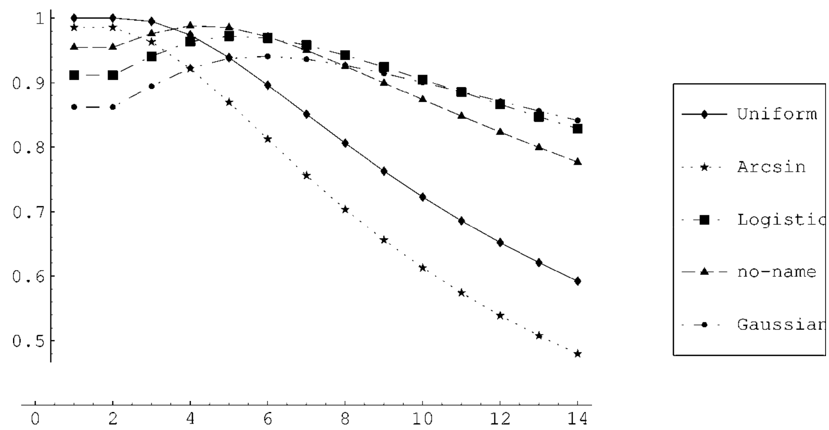

Figure 2: Efficiences of tests for different Maesono statistic

the efficiency decreases in a monotonic way.

Figure 2 represents similar picture for Maesono statistics. It is seen again that for uniform and arcsine distributions the plots are monotonic while for other distributions the maximum of efficiency is attained for various orders of Maesono statistics.

4 Conditions of Local Optimality

As is well known (see Bahadur (1967), Nikitin (1984) and Nikitin (1995) the local asymptotic optimality (LAO) of a sequence of statistics in Bahadur sense means the asymptotic equivalence of the local exact slope and of $2K(f,\theta)$ as $\theta \to 0$. Under regularity conditions described in Section 2 it means that for any sequence of test statistics $\{T_n\}$ one should have

$$l(T, f) = 4f^2(0) \int_{R^1} x^2 f(x)dx. \tag{17}$$

We are interested in those densities $f \in \mathcal{F}$ when (17) is true; such densities form the so-called domain of LAO in $\mathcal{F}$. The study of this "inverse" problem was started by Nikitin (1984), even if the importance of exploring the conditions of maximal Bahadur efficiency was already underlined by Savage (1969).

In case of the sign statistic we have $l(D, f) = 4f^2(0)(\mathbf{E}|X_1|)^2$, hence the condition of LAO (17) reduces to the condition $Var|X_1| = 0$ which cannot take place in $\mathcal{F}$. Hence the domain of LAO (we use here the terminology of (Nikitin , 1995, Chap. 6)) of S_n is empty in $\mathcal{F}$. Same result is true for the Kolmogorov goodness-of-fit statistic as shown in Durio and Nikitin (2001a).

More interesting is the case of general signed-rank statistics. Let consider only continuous and monotone score functions J. From (13) by Cauchy-Schwarz inequality and (10) we get

$$\{\int_0^\infty (J(2F(x) - 1)xdF(x)\}^2 \le \int_0^1 J^2(2x - 1)dx \cdot \int_0^\infty x^2 dF(x) = VarX_1.$$

Consequently the equality takes place on the set $\{x : f(x) \neq 0\}$ iff

$$J(2F(x) - 1) = C_3 x, \quad 0 \le x \le \infty,$$

where the solution on $(-\infty, 0)$ is defined by symmetry. Due to the properties of J we can always define the inverse function $J^{-1}(t)$ and hence on the support of f one has

$$F(t) = \frac{1}{2}(1 + J^{-1}(C_3 \cdot t)), \quad 0 \le t \le +\infty. \tag{18}$$

In case of linear score function $J_1(u) = \sqrt{12}u$ which corresponds to the Wilcoxon test we get as a solution the symmetric uniform distribution. We can consider this result as a characterization of the symmetric uniform distribution in $\mathcal{F}$.

We remark that the local optimality of the same statistic under the location alternative takes place for an entirely different distribution, namely logistic distribution (Nikitin , 1995, Chap. 6) that emphasizes the difference between the two types of alternatives. Moreover, it can be derived from Theorem 6.2 of Karlin and Studden (1966) that the efficiency of W_n with respect to the locally most powerful parametric test (for skew alternative it is the sample mean) can be arbitrary low. The same result in the case of location alternative is the celebrated result of Hodges and Lehmann (1956) saying that the lower bound of Pitman (or Bahadur) efficiency of Wilcoxon test with respect to the Student test is always not smaller than 0.864. The observed difference emphasizes once again the difference between the location and skew alternatives. Similar solutions of "truncated Pareto" type appear for "power" score functions.

In the case of "trigonometric "score function we arrive to the symmetric arc-sine law with the density

$$f(x) = \left(\pi\sqrt{C_4^2 - x^2}\right)^{-1} \mathbf{1}\{-C_4 \le x \le C_4\}.$$

We know from section 2 that this arcsine density satisfies all necessary regularity conditions. It is worth to mention that we got a new characterization of the arcsine density in the class $\mathcal{F}$ by the LAO property of linear signed-rank statistic with "trigonometric" scores under the skew alternative. Other characterizations of this distribution exist (see, e.g., Norton (1978) and Shantaram (1978)) but are very rare.

In the case of Maesono statistic of order r we use the alternative representation of the local index given in (15). From (16) we get again by Cauchy-Schwarz inequality

$$\begin{aligned}&(\textstyle\int_{R^1}[F^{r-1}(x) - (1-F(x))^{r-1}]xf(x)dx)^2\\ &\le \textstyle\int_{R^1}[F^{r-1}(x) - (1-F(x))^{r-1}]^2 dF(x)\cdot VarX_1 = r^2\sigma_r^2\cdot VarX_1.\end{aligned}$$

It follows from this formula and (16) that the LAO condition on the set $\{x : f(x) \neq 0\}$ results in

$$F^{r-1}(x) - (1-F(x))^{r-1} = C_5 x. \qquad (19)$$

The left-hand side of this equation is monotonic in x and hence the solution of (19) in distribution functions exists though it cannot be expressed for any r explicitly. The existence of this solution for which the Maesono statistics are locally optimal in Bahadur sense increases the interest to such statistics. However for $r = 2, 3$ we get the same solution, namely the symmetric uniform distribution.

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