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Insurance Premia Consistent with the Market*

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Abstract

We consider insurance prices in presence of an incomplete and competitive market. We show that if the insurance price system is internal, sublinear, and consistent with the market, then insurance prices are the maxima of their expected payments with respect to a family of risk neutral probabilities. We also show that under a simple additional assumption it is possible to decompose the obtained price in net premium plus safety loading.

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1 Introduction

In this paper we consider insurance prices (*premia*) in presence of an incomplete and competitive market for financial securities. Because of market incompleteness, while some insurance contracts can be replicated by financial portfolios traded on the market, in general they cannot.

Given a set $\mathcal{X}$ of insurance contracts, our purpose is to identify and study the consequences of some simple conditions on the insurance price functional $H : \mathcal{X} \to \mathbb{R}$ that we expect to be satisfied by any price schedule arising from the interaction of rational agents in this incomplete market structure. Our first result, Theorem 1, says that under these conditions there exists a unique compact and convex set $\mathbb{Q}$ of probabilities and a unique discount factor v such that

$$H(X) = v \sup_{Q \in \mathbb{Q}} \int_\Omega X dQ \tag{1}$$

for all insurance contracts X, and such that

$$H(X) = V(X) = v \int_\Omega X dQ \tag{2}$$

for all $Q \in \mathbb{Q}$ and all insurance contracts X replicable on the market, where $V(X)$ is the market price of the asset that replicates X.

Eq. (2) guarantees that all the probabilities in $\mathbb{Q}$ are risk neutral, that is, consistent with the market. On the other hand, the non-singleton nature of the set $\mathbb{Q}$, the key feature of Eq. (1), reflects the ambiguity due to the market imperfections, whose presence limits the information otherwise available to agents.

In Theorems 3 and 4 we refine the result by decomposing the price $H(X)$ in two parts, net premium and safety loading, thus generalizing to our setting a decomposition common in the insurance literature. Specifically, we show that, under an additional natural assumption, Eq. (1) becomes

$$H(X) = v \left[\int_\Omega X dP + \frac{1}{2} \operatorname{Amb}_{\mathbb{Q}}(X) \right]$$

where P is a base probability in the set $\mathbb{Q}$ and $\operatorname{Amb}_{\mathbb{Q}}(X)$ is a suitable measure of the ambiguity level of contract X (see Definition 2).

Mathematically, our results extend the decision-theoretic works of Gilboa and Schmeidler (1989), Siniscalchi (2000), and are related to recent works of Chateauneuf, Kast, and Lapied (1996), Wang, Young, and Panjer (1997), Artzner, Delbaen, Eber, and Heath (1999), Delbaen (2000). Our contribution is threefold: (i) We consider market consistency, an obviously important issue in pricing insurances. (ii) We provide a decomposition of the price functional in net premium and safety loading, a convenient representation for insurance purposes. (iii) We consider general, possibly unbounded, contracts on possibly infinite state spaces; this is achieved by only imposing weak structural requirements on the sets of contracts, and without assuming the existence of an *a priori* probability on Ω.

The paper is organized as follows. Section 2 lists and discusses the requirements on the markets and Section 3 on the price functionals. Section 4 is devoted to results, while all proofs are relegated to the Appendix (Section 6), after the final remarks of Section 5.

2 Markets

We consider one period of uncertainty. At date 0 agents buy financial contracts in a competitive and frictionless incomplete market and insurance contracts from insurance firms, for consumption at time T. Let Ω be the state space and $\mathfrak{F}$ the event σ-field at time T. A contract, financial or insurance, is described by a random variable $X : \Omega \to \mathbb{R}$, where $X(\omega)$ represents its payoff at time T if state ω obtains. We denote by $\mathcal{B}_0(\Omega, \mathfrak{F})$ the set of all simple random variables, by $\mathcal{B}(\Omega, \mathfrak{F})$ the set of all bounded random variables, and by $\mathcal{M}(\Omega, \mathfrak{F})$ the set of all bounded below random variables. If X is a random variable and β is a real number, the random variable $X \wedge \beta$, defined by

$$(X \wedge \beta)(\omega) = \min\{X(\omega), \beta\}$$

for $\omega \in \Omega$, represents a stop-loss contract providing a coverage $X(\omega)$ if $X(\omega) < \beta$ and β otherwise.

Let $\mathcal{X}$ be the set of all insurance contracts. Usually, such contracts are represented by nonnegative, possibly unbounded, random variables; that is, $\mathcal{X} \subseteq \mathcal{M}^+(\Omega, \mathfrak{F})$ (see assumption S.1 below). However, when surrenders are taken into account, insurance firms have to price also nonpositive random variables. Moreover, they may want to sell X_1 and buy back X_2 at the same time, and consequently they may have to deal with the real-valued random variable $X_1 - X_2$. For these reasons, we will also consider the case $\mathcal{X} \subseteq \mathcal{M}(\Omega, \mathfrak{F})$ (see assumption S.2 below).

Let $\mathcal{Y}$ be the set of the payoffs of all possible portfolios of financial securities traded on the market. We assume that the riskless bond 1_Ω, which pays 1 in each state of the world, is traded on the market, and that each element of $\mathcal{Y}$ is bounded.[1] An insurance contract is *replicable* if there exists a portfolio of financial securities with the same payoffs as the contract.

Specifically, we will make one of the two following assumptions on the market structure:

S.1 $\mathcal{Y}$ is a subset of $\mathcal{B}(\Omega, \mathfrak{F})$ containing 1_Ω and such that $Y + 1_\Omega \in \mathcal{Y}$ for all $Y \in \mathcal{Y}$, $X \wedge n \in \mathcal{X}$ for all $X \in \mathcal{X}$ and all $n \geq 1$, and

$$\langle \mathcal{Y} \rangle^+ + \mathcal{B}_0^+(\Omega, \mathfrak{F}) \subseteq \mathcal{X} \subseteq \mathcal{M}^+(\Omega, \mathfrak{F}).$$

S.2 $\mathcal{Y}$ is a subset of $\mathcal{B}(\Omega, \mathfrak{F})$ containing 1_Ω, $X \wedge n \in \mathcal{X}$ for all $X \in \mathcal{X}$ and all $n \geq 1$, and

$$\langle \mathcal{Y} \rangle + \mathcal{B}_0(\Omega, \mathfrak{F}) \subseteq \mathcal{X} \subseteq \mathcal{M}(\Omega, \mathfrak{F}).$$

[1] See Remark 6 in Section 5.

The first assumption is in line with the classic approach, in which insurance contracts are nonnegative random variables, while the second one allows for surrenders and buy backs too.

More importantly, assumptions S.1 and S.2 do not require neither $\mathcal{Y}$ nor $\mathcal{X}$ to be vector spaces, thus making it possible to model markets with trading constraints (e.g., financial markets with selling limitations).

Notice that these assumptions are satisfied in some popular settings. For example, S.1 clearly holds if $\mathcal{Y}$ is a convex cone in $\mathcal{B}(\Omega, \mathfrak{F})$ containing 1_Ω and

$$\mathcal{X} = \mathcal{B}^+(\Omega, \mathfrak{F}) \text{ or } \mathcal{X} = \mathcal{M}^+(\Omega, \mathfrak{F});$$

while S.2 holds if $\mathcal{Y}$ is a subset of $\mathcal{B}(\Omega, \mathfrak{F})$ containing 1_Ω and

$$\mathcal{X} = \mathcal{B}(\Omega, \mathfrak{F}) \text{ or } \mathcal{X} = \mathcal{M}(\Omega, \mathfrak{F}).$$

3 Prices

In this section we present and discuss some properties of insurance policies' prices that we expect to be satisfied by any price schedule arising from the interaction of rational agents. These properties are based on few economic assumptions:

- competition between insurance firms pares policies' prices, so that, without loss of generality, we can focus on a single insurance firm;
- no arbitrage opportunities are left to customers;
- the financial market is frictionless and without arbitrage opportunities, that is, there exists a finitely additive probability Q such that the market price functional $V : \mathcal{Y} \to \mathbb{R}$ is given by

$$V(Y) = v \int_\Omega Y dQ \tag{3}$$

  for all $Y \in \mathcal{Y}$ (a probability Q such that Eq. (3) holds is called *risk neutral* or *risk adjusted*). Since it causes no loss of generality, we assume $v = V(1_\Omega) = 1$.[2]

Let $H : \mathcal{X} \to \mathbb{R} \cup \{\infty\}$ be the insurance price functional in terms of consumption at date 0; $H(X)$ is the price for insurance contract X, that is, the price at which the firm sells it to customers (when $H(X)$ is infinite, it simply means that the firm will not cover X). We now present and discuss some desirable properties of the price schedule H, based on the economic assumptions we just spelled out. We start with three standard properties.

A.1 *Internality*: $H(X) \leq \sup_{\omega \in \Omega} X(\omega)$ for all $X \in \mathcal{X}$.

[2] See Remark 5 in Section 5.

This is a natural price condition: in presence of a financial market, if it held $H(X) > \sup X$, any customer needing a coverage of X would buy $\sup X$ units of the riskless bond rather than the firm's policy.

A.2 *Positive homogeneity*: $H(\alpha X) = \alpha H(X)$ for all $\alpha \in [0, 1]$ and all $X \in \mathcal{X}$ such that $\alpha X \in \mathcal{X}$.

According to A.2, premia are independent of the particular currency in which they are denominated. In addition, there are no transaction costs for proportional coverages, a liquidity requirement on the firm.

A.3 *Subadditivity*: $H(X_1 + X_2) \leq H(X_1) + H(X_2)$ for all $X_1, X_2 \in \mathcal{X}$ such that $X_1 + X_2 \in \mathcal{X}$.

This is another natural price condition: if for some X_1 and X_2 it held $H(X_1 + X_2) > H(X_1) + H(X_2)$, a customer would buy X_1 and X_2 separately rather than $X_1 + X_2$.

Notice that, under A.1 and A.2, subadditivity implies that

$$H(X) + H(-X) \geq 0 \text{ and } \inf X \leq H(X) \leq \sup X, \tag{4}$$

for all $X \in \mathcal{X}$ such that $-X \in \mathcal{X}$. As buying X back is the same as selling $-X$, $-H(-X)$ is simply the surrender value for X and so Eq. (4) means that the net balance of the firm for surrenders is nonnegative and no arbitrage opportunity is left to customers, who cannot form at a negative cost a portfolio of insurance contracts with nonnegative payoffs.[3]

A functional H satisfying A.2 and A.3 is called *sublinear*. Internality and sublinearity are standard assumptions for insurance prices. The next assumption is, in contrast, peculiar to our analysis.

A.4 *Consistency with the market*: $H(X + Y) = H(X) + V(Y)$ for all $X \in \mathcal{X}$ and all $Y \in \mathcal{Y}$ such that $X + Y \in \mathcal{X}$.

According to A.4, whenever part of an insurance contract can be replicated by a portfolio of financial securities, the price of that part equals the market price of the replicating portfolio. This is a reasonable price assumption: if it held $H(X + Y) > H(X) + V(Y)$, customers needing a coverage of $X + Y$ would buy insurance X from the firm and the portfolio Y on the market. On the other hand, if it held $H(X + Y) < H(X) + V(Y)$, customers needing a coverage of X would buy the insurance $X + Y$ and sell Y on the market, thus paying $H(X + Y) - V(Y) < H(X)$. Notice that A.4 and A.2 imply that $H(Y) = V(Y)$ for all $Y \in \mathcal{Y} \cap \mathcal{X}$.

We conclude with two continuity axioms. The first one allows us to deal with unbounded contracts, and it requires that the price of any contract can be obtained by approximation from below. The axiom is trivially satisfied if $\mathcal{X}$ consists of bounded random variables.

[3] Although surrenders may be seen as arbitrage opportunities for the firms, notice that the decisions to buy insurance contracts, or to sell them back, are taken by agents (and not by firms).

A.5 *Subcontinuity*: $H(X) = \lim_n H(X \wedge n)$ for all $X \in \mathcal{X}$ such that eventually $X \wedge n \in \mathcal{X}$.

The next axiom will ensure the countable additivity of the probabilities involved in Theorem 1.

A.6 *Outer Continuity*: if $\{X_n\}_{n\geq 1} \subseteq \mathcal{B}_0^+(\Omega, \mathfrak{F}) \cap \mathcal{X}$ is such that $X_n \downarrow X \in \mathcal{X}$, then $\lim_n H(X_n) = H(X)$.

4 Results

4.1 Representation

We can now state our pricing result. In reading it, recall that $\mathcal{X}$ is not required to be a vector space and that the price of an unbounded insurance contract can well be infinite.

Theorem 1 *Let the market structure satisfy either S.1 or S.2. Then, the following statements are equivalent:*

(i) The functional $H : \mathcal{X} \to \mathbb{R} \cup \{\infty\}$ satisfies A.1-A.6.

(ii) There exists a unique weak compact and convex set $\mathbb{Q}$ of countably additive and risk neutral probabilities such that, for all $X \in \mathcal{X}$,

$$H(X) = \sup_{Q \in \mathbb{Q}} \int_\Omega X dQ. \tag{5}$$

Remark. For each bounded $X \in \mathcal{X}$ the sup appearing in Eq. (5) is a max.

In other words, under assumptions A.1-A.6, premia are fixed by considering the maximum expected loss relative to a family $\mathbb{Q}$ of risk neutral probabilities. The non-singleton nature of $\mathbb{Q}$ reflects the ambiguity due to the limited information that consumers and firms have in dealing with contracts that are not replicable by the market. The fact that, in general, $\mathbb{Q}$ is a proper subset of the set of all risk neutral probabilities reflects some form of market power of firms, for example due to some better information that they may have based on historical records, sector studies, and so on.

Mathematically, Theorem 1 extends a result proved by Gilboa and Schmeidler (1989) in a decision-theoretic setting, and it is related to some recent results of Chateauneuf, Kast, and Lapied (1996), Wang, Young, and Panjer (1997), Artzner, Delbaen, Eber, and Heath (1999), Delbaen (2000). Relative to these papers, Theorem 1 has two main novel features: (a) it allows market consistency, and (b) it considers contracts on a general measurable space, under weak structural assumptions and without assuming the existence of *a priori* probability on Ω, an *ad hoc* assumption which seems difficult to interpret. In our result, probabilities pop up directly from market properties.

4.2 Decomposition

We now refine the representation in Theorem 1. In the insurance literature it is usual to decompose the price $H(X)$ in two parts, the *net premium* – that is, the expected payment $E(X)$ of X – and the *safety loading*, which, in some sense, measures the riskiness of X. For example, the *standard deviation principle* suggests

$$H(X) = E(X) + \gamma\sqrt{\mathrm{Var}(X)}$$

where γ is a positive constant, while the *variance principle,* the *standard semi-deviation principle,* and the *semi-variance principle* are similar with respectively $\mathrm{Var}(X)$, $\sqrt{\mathrm{SemiVar}(X)}$, and $\mathrm{SemiVar}(X)$ in place of $\sqrt{\mathrm{Var}(X)}$.

Under the following assumption, inspired by Siniscalchi (2000), it is possible to provide a decomposition of this type in our setting.

A.7 *Symmetry:* for all $X_1, X_2 \in \mathcal{X}$ such that $-X_1, -X_2$, $X_1 + X_2$, and $-(X_1 + X_2)$ belong to $\mathcal{X}$, we have

$$H(X_1)+H(X_2)+H(-(X_1+X_2)) = H(X_1+X_2)+H(-X_1)+H(-X_2).$$

According to this assumption, on the firm's net balance it has the same effect either to sell X_1 and X_2 separately and then buy them back together, or to sell X_1 and X_2 together and buy them back separately. As the surrender value for X is $-H(-X)$, this can be paraphrased as follows: the net balance of the firm for surrenders does not depend on the decomposition of the contracts. For example, the standard deviation and the variance principles are both symmetric, while the standard semi-deviation and semivariance principles are not.

To state the result, we introduce a notion of safety loading appropriate for our setting.

Definition 2 *The* ambiguity level *of a contract X with respect to a set $\mathbb{Q}$ of probabilities is*

$$\mathrm{Amb}_{\mathbb{Q}}(X) = \sup_{Q,Q' \in \mathbb{Q}} \left| \int_\Omega X dQ - \int_\Omega X dQ' \right|,$$

with the convention $\infty - \infty = 0$.

The nonsingleton nature of $\mathbb{Q}$ reflects the limited information due to the markets' incompleteness, and $\mathrm{Amb}_{\mathbb{Q}}(X)$ measures the impact of this limited information on the contract X. The higher $\mathrm{Amb}_{\mathbb{Q}}(X)$ is, the more limited is the information on which the pricing of X is based.

Theorem 3 *Let the market structure satisfy S.2. The following statements are equivalent:*

(i) The functional $H : \mathcal{X} \to \mathbb{R} \cup \{\infty\}$ satisfies A.1–A.7.

(ii) There exists a unique weak compact, symmetric, and convex set $\mathbb{Q}$ of countably additive and risk neutral probabilities, and a unique $P \in \mathbb{Q}$ such that

$$H(X) = \int_\Omega X dP + \frac{1}{2} \operatorname{Amb}_{\mathbb{Q}}(X) \tag{6}$$

for all $X \in \mathcal{X}$, where $P = \frac{1}{2} \max_{Q \in \mathbb{Q}} Q + \frac{1}{2} \min_{Q \in \mathbb{Q}} Q$ is the center of symmetry of $\mathbb{Q}$.[4]

In other words, a family $\mathbb{Q}$ of risk neutral probabilities is considered and one of them is selected by fair weighting. Insurance contracts are then priced by their net premia with respect to P, plus a safety loading based on the ambiguity level of X.

Theorem 3 extends to our insurance setting a recent representation result proved by Siniscalchi (2000) in the decision-theoretic setting of Gilboa and Schmeidler (1989). Besides market consistency and unbounded contracts, the contribution of Theorem 3 is the term $\operatorname{Amb}_{\mathbb{Q}}(X)$, which in our insurance setting can be interpreted as a safety loading, but that, more generally, is a natural ambiguity level.

Example. Let $\mathbb{Q} = \{\alpha Q_1 + (1-\alpha) Q_2 : \alpha \in [0,1]\}$, that is, suppose that Q_1 and Q_2 are the relevant countably additive and risk neutral probabilities. The set $\mathbb{Q}$ is weak compact, symmetric, and convex with center of symmetry $P = (Q_1 + Q_2)/2$. Since $\operatorname{Amb}_{\mathbb{Q}}(X) = \left|\int_\Omega X dQ_1 - \int_\Omega X dQ_2\right|$, by Theorem 3, the following pricing rule is compatible with A.1-A.7:

$$H(X) = \int_\Omega X d\left(\frac{Q_1 + Q_2}{2}\right) + \frac{1}{2}\left|\int_\Omega X dQ_1 - \int_\Omega X dQ_2\right|,$$

that is,

$$H(X) = \max\left\{\int_\Omega X dQ_1, \int_\Omega X dQ_2\right\}.$$

Notice that for the set $\mathbb{Q}$ of the example

$$\begin{aligned} \left|\int_\Omega X dQ_1 - \int_\Omega X dQ_2\right| &= \frac{1}{2}\left|\int_\Omega X d(Q_1 - Q_2)\right| + \frac{1}{2}\left|\int_\Omega X d(Q_2 - Q_1)\right| \\ &= \int_{\partial(\mathbb{Q}-\mathbb{Q})} \left|\int_\Omega X d\mu\right| d\left(\frac{1}{2}\delta_{Q_1 - Q_2}(\mu) + \frac{1}{2}\delta_{Q_2 - Q_1}(\mu)\right), \end{aligned}$$

and so the ambiguity level of a contract X can be written as

$$\operatorname{Amb}_{\mathbb{Q}}(X) = \int_{\partial(\mathbb{Q}-\mathbb{Q})} \left|\int_\Omega X d\mu\right| dm(\mu),$$

[4] The set $\mathbb{Q}$ is symmetric if there exists a probability P, the center of symmetry, such that $(2P - Q) \in \mathbb{Q}$ for all $Q \in \mathbb{Q}$.

where m is an even Borel measure over the relative boundary $\partial(\mathbb{Q}-\mathbb{Q})$ of $\mathbb{Q}-\mathbb{Q}$. In other words, we can view the ambiguity level as the result of firms' subjective weighting over the spreads $\mathbb{Q}-\mathbb{Q}$, with the weighting given by the measure m.

It turns out that, by using some results in convex geometry, we can establish the same characterization of the ambiguity level for generalized zonoids, an important family of finite dimensional symmetric sets. Specifically, a *zonoid* is a limit of finite sums of line segments (the zonotopes). If there exist zonoids $\mathbb{Q}_1$ and $\mathbb{Q}_2$ such that $\mathbb{Q}_1 = \mathbb{Q} + \mathbb{Q}_2$, then $\mathbb{Q}$ is a *generalized zonoid* (see Appendix 6.1.1 for more details).

Theorem 4 *Let $\mathbb{Q}$ be a (generalized) zonoid of risk neutral probabilities. Then there exists a unique even (signed) measure m over the Borel σ-algebra of $\partial(\mathbb{Q}-\mathbb{Q})$ such that, for each $X \in \mathcal{B}(\Omega, \mathfrak{F})$,*

$$\mathrm{Amb}_{\mathbb{Q}}(X) = \int_{\partial(\mathbb{Q}-\mathbb{Q})} \left| \int_{\Omega} X d\mu \right| dm(\mu).$$

An extension of this result to all finite dimensional symmetric sets $\mathbb{Q}$ can be obtained by replacing the even measure m with a suitable even distribution, along the lines of Weil (1976) (see Appendix 6.1.1).

5 Concluding Remarks

1. Next we provide a finitely additive version of Theorem 3, where we drop the outer continuity axiom A.6.

Proposition 5 *Suppose either S.1 and $\mathcal{B}^+(\Omega, \mathfrak{F}) \subseteq \mathcal{X}$ or S.2 and $\mathcal{B}(\Omega, \mathfrak{F}) \subseteq \mathcal{X}$. The following statements are equivalent:*

(i) The functional $H : \mathcal{X} \to \mathbb{R} \cup \{\infty\}$ satisfies A.1-A.5.

(ii) There exists a unique weak compact and convex set $\mathbb{Q}$ of finitely additive risk neutral probabilities such that, for all $X \in \mathcal{X}$,*

$$H(X) = \sup_{Q \in \mathbb{Q}} \int_{\Omega} X dQ.$$

In addition, if S.2, $\mathcal{B}(\Omega, \mathfrak{F}) \subseteq \mathcal{X}$, and A.7 hold, $\mathbb{Q}$ is symmetric, weak compact, and there exists a unique $P \in \mathbb{Q}$ such that, for all $X \in \mathcal{X}$,

$$H(X) = \int_{\Omega} X dP + \frac{1}{2} \mathrm{Amb}_{\mathbb{Q}}(X), \tag{7}$$

where $P = \frac{1}{2} \max_{Q \in \mathbb{Q}} Q + \frac{1}{2} \min_{Q \in \mathbb{Q}} Q$ is the center of symmetry of $\mathbb{Q}$.

Notice that the set $\mathbb{Q}$ in Eq. (7) is weak compact and not just weak* compact. This is a significant improvement because, by the classic Eberlein-Smulian Theorem, $\mathbb{Q}$ is also sequentially compact, a very convenient property.

2. Theorem 1 turns out to be fully in the spirit of an old work of de Finetti and Obry (1933), who write: "...consideriamo contemporaneamente un certo numero di tariffari additivi e determiniamo il prezzo dei singoli stati calcolando secondo il tariffario che per ciascuno di essi risulta più gravoso: con questo procedimento otteniamo il più generale sistema di prezzi che soddisfi la ...[positiva omogeneità]... e consenta il criterio ...[di subadditività]..."[5]

3. All our results remain true if, instead of internality, we assumed monotonicity, that is, $X_1 \geq X_2 \Rightarrow H(X_1) \geq H(X_2)$.

4. If $H(X_1 + X_2) = H(X_1) + H(X_2)$ for all comonotonic $X_1, X_2 \in \mathcal{X}$ with $X_1 + X_2 \in \mathcal{X}$, then Eq. (5) becomes $H(X) = \int_\Omega X dC$, where $\int_\Omega X dC$ is the Choquet expected value of X with respect to $C(\cdot) = \max_{Q \in \mathbb{Q}} Q(\cdot)$. This is consistent with the axiomatization of Wang, Young, and Panjer (1997).

5. If $0 < V(1_\Omega) = v \neq 1$ everything still works, with some obvious changes. For example, Eq. (5) becomes $H(X) = v \sup_{Q \in \mathbb{Q}} \int_\Omega X dQ$.

6. If the riskless bond is *not* traded on the market, the discount factor is no longer fixed. Up to some minor changes, our analysis still holds. For example, Eq. (5) becomes $H(X) = \sup_{Q \in \mathbb{Q}} v_Q \int_\Omega X dQ$.

6 Appendix: proofs and related material

The proof of the main theorem relies on some lemmas, which might be of some interest on their own. We state and prove them, and we then move to the proofs of the results in subsection 6.4.

6.1 Envelopes of linear functionals

Let $(E, \geq)$ be an ordered vector space and U a nonempty subset of E. We denote by $\langle U \rangle$ the span of U in E.

A functional I from E to $\mathbb{R}$ is said to be U*-additive* if $I(x+u) = I(x) + I(u)$ for all $x \in E$ and all $u \in U$. We denote by E' the set of all linear functionals on E, and if $f \in E'$ we write $f(x)$ or $\langle x, f \rangle$ indifferently. If (E, τ) is a topological vector space, E^* denotes the subset of E' consisting of τ-continuous functionals.

Next lemma extends a simple result proved in Marinacci (1997).

[5] "...let us consider a given number of additive tariffs and determine the price of each position by taking the highest: in this way we obtain the most general price system satisfying ...[positive homogeneity]... and ...[subadditivity]..."

Lemma 6 *A functional $I : E \to \mathbb{R}$ is sublinear, U-additive and such that $I(x) \le 0$ for all $x \le 0$ if and only if there exists a unique convex and $\sigma(E', E)$-compact set K of linear and monotone functionals on E, with $f_{|\langle U\rangle} = I_{|\langle U\rangle}$ for all $f \in K$, and such that*

$$I(x) = \max_{f\in K} f(x)$$

for all $x \in E$. Moreover, if (E, τ) is a topological vector space and I is τ-continuous, all functionals $f \in K$ are τ-continuous.

Proof. For all $u \in U$ and all $x \in E$, $I(x) = I(x - u + u) = I(x - u) + I(u)$, that is $I(x - u) = I(x) - I(u)$; if $\alpha > 0$, $I(x + \alpha u) = \alpha\left(I\left(\frac{x}{\alpha} + u\right)\right) = I(x) + \alpha I(u)$; if $\alpha < 0$, $I(x + \alpha u) = I(x - (-\alpha)u) = -\alpha\left(I\left(\frac{x}{-\alpha} - u\right)\right) = I(x) + \alpha I(u)$. So that, for all $x \in X$ and all $\sum_{i=1}^{n} \alpha_i u_i \in \langle U\rangle$ (with $\alpha_i \in \mathbb{R}$ and $u_i \in U$) we have

$$I\left(x + \sum_{i=1}^{n} \alpha_i u_i\right) = I(x) + \sum_{i=1}^{n} \alpha_i I(u_i),$$

and choosing $x = 0$,

$$I\left(\sum_{i=1}^{n} \alpha_i u_i\right) = \sum_{i=1}^{n} \alpha_i I(u_i),$$

whence

$$I\left(x + \sum_{i=1}^{n} \alpha_i u_i\right) = I(x) + I\left(\sum_{i=1}^{n} \alpha_i u_i\right),$$

and I is $\langle U\rangle$-addititive. W.l.o.g., we can assume that U is a subspace of E.

Set

$$K = \{f \in E' : f(x) \le I(x) \text{ for all } x \in E \text{ and } f(u) = I(u) \text{ for all } u \in U\}.$$

Clearly K is convex and $\sigma(E', E)$-compact. If $x \ge 0$, then $-x \le 0$, $f(-x) \le I(-x) \le 0$ and $f(x) \ge 0$. Thus K consists of positive linear functionals. Choose $\widehat{x} \in E$ and consider the following two cases.

1. If $\widehat{x} \in E \backslash U$. Set $E_{\widehat{x}} = \{\alpha\widehat{x} + u : \alpha \in \mathbb{R}, u \in U\}$. Let $f_{\widehat{x}}$ be the linear functional on $E_{\widehat{x}}$ defined by

   $$f_{\widehat{x}}(\alpha\widehat{x} + u) = \alpha I(\widehat{x}) + I(u).$$

   We have $f_{\widehat{x}}(\alpha\widehat{x} + u) = I(\alpha\widehat{x} + u)$ if $\alpha \ge 0$, while, if $\alpha < 0$, $f_{\widehat{x}}(\alpha\widehat{x} + u) = -I(-\alpha\widehat{x} - u) \le I(\alpha\widehat{x} + u)$. Thus, $f_{\widehat{x}}(x) \le I(x)$ for all $x \in E_{\widehat{x}}$.

2. Else $\widehat{x} \in U$. Set $E_{\widehat{x}} = U$ and $f_{\widehat{x}}(x) = I(x)$ for all $x \in E_{\widehat{x}}$. Clearly, $f_{\widehat{x}}(x) \le I(x)$ for all $x \in E_{\widehat{x}}$.

In both cases we have $f_{\widehat{x}}(x) \leq I(x)$ for all $x \in E_{\widehat{x}}$. By the Hahn-Banach Theorem there exists a linear functional $\widehat{f}$ on E such that $\widehat{f}(x) = f_{\widehat{x}}(x)$ for all $x \in E_{\widehat{x}}$, and $\widehat{f}(x) \leq I(x)$ for all $x \in E$. Therefore, $\widehat{f} \in K$, and

$$I(\widehat{x}) = \widehat{f}(\widehat{x}) = \max_{f \in K} f(\widehat{x}).$$

Suppose that $I(\cdot) = \max_{g \in C} g(\cdot)$ for another convex and $\sigma(E', E)$-compact set C of linear functionals on E. W.l.o.g., there exists $g \in C \backslash K$; by a standard separation result (see Dunford and Schwartz, 1958, Theorem V.2.10), there exists $x \in E$, $c \in \mathbb{R}$, and $\varepsilon > 0$ such that $g(x) \geq c > c - \varepsilon \geq f(x)$ for all $f \in K$. Hence $g(x) > \max_{f \in K} f(x)$, a contradiction.

Finally, if $f \in K$ and $\{x_j\}$ is a net in E such that $x_j \xrightarrow{\tau} 0$, then

$$-I(-x_j) \leq -f(-x_j) = f(x_j) \leq I(x_j),$$

and hence $f(x_j) \to 0$ whenever I is τ-continuous. ■

Lemma 7 *Let K be a convex and $\sigma(E', E)$-compact subset of E' and let*

$$I(x) = \max_{f \in K} \langle x, f \rangle$$

for all $x \in E$. The functional $I(x) - I(-x)$ is linear if and only if K is symmetric. In this case:

1. *The only center of symmetry of K is the functional*

$$\widehat{f}(x) = \frac{1}{2}(I(x) - I(-x)) \quad \forall x \in E.$$

2. *The following identities hold*

$$I(x) = \widehat{f}(x) + \frac{1}{2} \max_{f, g \in K} |\langle x, f - g \rangle| = \widehat{f}(x) + \max_{f \in K} \left| \left\langle x, \widehat{f} - f \right\rangle \right| \quad \forall x \in E.$$

Proof. Let K be symmetric and $\widehat{f}$ be a center of symmetry for K. Let $x \in E$,

$$I(x) - I(-x) = \max_{f \in K} \langle x, f \rangle - \max_{g \in K} \langle -x, g \rangle = \max_{f \in K} \langle x, f \rangle + \min_{g \in K} \langle x, g \rangle.$$

Let $f^* \in \operatorname{argmax}_{f \in K} \langle x, f \rangle$, and let $f_* = 2\widehat{f} - f^*$. If $f_* \notin \operatorname{argmin}_{g \in K} \langle x, g \rangle$, there exists $g_* \in K$ such that $\langle x, g_* \rangle < \langle x, f_* \rangle$. Hence,

$$\left\langle x, 2\widehat{f} - g_* \right\rangle = \left\langle x, 2\widehat{f} \right\rangle - \langle x, g_* \rangle > \left\langle x, 2\widehat{f} \right\rangle - \langle x, f_* \rangle = \left\langle x, 2\widehat{f} - f_* \right\rangle = \langle x, f^* \rangle,$$

which is absurd, since $2\widehat{f} - g_* \in K$. Therefore,

$$I(x) - I(-x) = \langle x, f^* \rangle + \langle x, f_* \rangle = \left\langle x, f^* + 2\widehat{f} - f^* \right\rangle = 2\left\langle x, \widehat{f} \right\rangle.$$

Hence $I(x) - I(-x)$ is linear and $\widehat{f}$ is unique.

Conversely, if $I(x) - I(-x)$ is linear, set $\widehat{f}(x) = \frac{1}{2}(I(x) - I(-x))$ for all $x \in E$. Let $f' \in K$, we have

$$\begin{aligned}\left\langle x, 2\widehat{f} - f' \right\rangle &= I(x) - I(-x) - \langle x, f' \rangle = \\ &= \max_{f \in K} \langle x, f \rangle + \min_{g \in K} \langle x, g \rangle - \langle x, f' \rangle \geq \min_{g \in K} \langle x, g \rangle\end{aligned}$$

for all $x \in E$. If $2\widehat{f} - f' \notin K$, by a standard separation result (see Dunford and Schwartz, 1958, Theorem V.2.10), there exists $x' \in E$, $c \in \mathbb{R}$, and $\varepsilon > 0$ such that $\left\langle x', 2\widehat{f} - f' \right\rangle \leq c - \varepsilon < c \leq \langle x', g \rangle$ for all $g \in K$. Hence, $\left\langle x', 2\widehat{f} - f' \right\rangle < \min_{g \in K} \langle x', g \rangle$, a contradiction. Therefore, $2\widehat{f} - f' \in K$ and $\widehat{f}$ is a center of symmetry for K.

Now,

$$\begin{aligned}I(x) &= \frac{1}{2}(I(x) - I(-x)) + \frac{1}{2}(I(x) + I(-x)) = \\ &= \left\langle x, \widehat{f} \right\rangle + \frac{1}{2}\left(\max_{f \in K} \langle x, f \rangle + \max_{g \in K} \langle x, -g \rangle \right) = \\ &= \left\langle x, \widehat{f} \right\rangle + \frac{1}{2} \max_{f, g \in K} \langle x, f - g \rangle = \\ &= \left\langle x, \widehat{f} \right\rangle + \frac{1}{2} \max_{f, g \in K} |\langle x, f - g \rangle| .\end{aligned}$$

In fact, letting $f^* \in \operatorname{argmax}_{f \in K} \langle x, f \rangle$, $g^* \in \operatorname{argmax}_{g \in K} \langle x, g \rangle$, we have:

$$\max_{f \in K} \langle x, f \rangle + \max_{g \in K} \langle x, -g \rangle = \langle x, f^* \rangle + \langle x, -g^* \rangle = \langle x, f^* - g^* \rangle \leq \max_{f, g \in K} \langle x, f - g \rangle .$$

The converse inequality is trivial.

Finally, notice that, for all $f \in K$,

$$\left| \left\langle x, \widehat{f} - f \right\rangle \right| = \left\langle x, \widehat{f} - f \right\rangle \text{ or } \left| \left\langle x, \widehat{f} - f \right\rangle \right| = \left\langle x, f - \widehat{f} \right\rangle = \left\langle x, \widehat{f} - \left(2\widehat{f} - f \right) \right\rangle$$

and hence $\max_{f \in K} \left| \left\langle x, \widehat{f} - f \right\rangle \right| = \max_{f \in K} \left\langle x, \widehat{f} - f \right\rangle$, thus

$$\begin{aligned}I(x) &= I(x) - I(-x) + I(-x) = \\ &= \left\langle x, \widehat{f} \right\rangle + \left\langle x, \widehat{f} \right\rangle + \max_{f \in K} \langle x, -f \rangle \\ &= \left\langle x, \widehat{f} \right\rangle + \max_{f \in K} \left\langle x, \widehat{f} - f \right\rangle = \\ &= \left\langle x, \widehat{f} \right\rangle + \max_{f \in K} \left| \left\langle x, \widehat{f} - f \right\rangle \right| .\end{aligned}$$

This concludes the proof. ■

6.1.1 Finite dimensional case

We will now consider the case in which E is a normed space and K is a nonempty, d-dimensional, convex, compact and symmetric subset of E^*, with center of symmetry 0 (and hence $K = -K$). Let F be the span of K. If K is a finite sum of line segments in F, all centered in the origin, it is a *zonotope*. If K is the limit (in the Hausdorff metric of F) of a sequence of zonotopes, it is a *zonoid*. If there exist zonoids K_1 and K_2 such that $K_1 = K + K_2$, K is a *generalized zonoid*. Any convex, compact subset of F which is symmetric about 0 belongs to the closure in the Hausdorff metric of generalized zonoids. If the center of symmetry of K is $\widehat{f} \neq 0$, we call K a *zonotope*, *zonoid*, *generalized zonoid* if $K - \widehat{f}$ is respectively a zonotope, zonoid, generalized zonoid.

Let $\{f_1, f_2, ..., f_d\}$ be a base for F. Then there exist $\{e_1, e_2, ..., e_d\} \subseteq E$ such that $\langle e_j, f_i \rangle = 1$ if $j = i$ and 0 otherwise. Letting $V = \langle e_1, e_2, ..., e_d \rangle$ and $F^\perp = \{x \in E : g(x) = 0 \, \forall g \in F\}$, $E = F^\perp \oplus V$. For all $x \in E$, denote by $v(x)^\perp + v(x)$ the decomposition of x on $F^\perp$ and V, and notice that, for all $g \in F$,

$$\langle x, g \rangle = \left\langle v(x)^\perp, g \right\rangle + \langle v(x), g \rangle = \langle v(x), g \rangle$$

and hence

$$I(x) = \max_{f \in K} \langle x, f \rangle = \max_{f \in K} \langle v(x), f \rangle = I(v(x)).$$

Consider the embedding

$$\begin{array}{ccc} V & \rightarrow & F \\ v & \mapsto & \widetilde{v} \end{array}$$

defined by $\widetilde{e_j} = f_j$, and the inner product on F defined by

$$\left(\sum_{j=1}^{d} \beta_j f_j, \sum_{i=1}^{d} \gamma_i f_i \right) = \sum_{j=1}^{d} \beta_j \gamma_j$$

induced by the natural embedding of F in $\mathbb{R}^d$. Notice that, for all $v = \sum_{j=1}^{d} \beta_j e_j \in V$ and all $g = \sum_{i=1}^{d} \gamma_i f_i \in F$,

$$\langle v, g \rangle = \left\langle \sum_{j=1}^{d} \beta_j e_j, \sum_{j=1}^{d} \gamma_i f_i \right\rangle = \sum_{j=1}^{d} \beta_j \gamma_j = \left(\sum_{j=1}^{d} \beta_j f_j, \sum_{i=1}^{d} \gamma_i f_i \right) = (\widetilde{v}, g).$$

Therefore

$$I(x) = I(v(x)) = \max_{f \in K} \langle v(x), f \rangle = \max_{f \in K} \left(\widetilde{v(x)}, f \right).$$

We will denote by $\widetilde{I} : F \rightarrow \mathbb{R}$ the functional

$$\widetilde{I}(g) = \max_{f \in K} (g, f).$$

Let K° and ∂K be the interior and boundary of K *relative* to F (notice that $0 \in K^\circ$), and by S the unit sphere in F in the norm $\|\cdot\|$ induced by the inner product.

Remark 1 *The projection of* ∂K *on* S *defined by* $N : f \mapsto \frac{f}{\|f\|}$ *is a homeomorphism.*

Remark 2 *By the change of variable formula, for any continuous and positively homogeneous function* $\phi : F \to \mathbb{R}$ *and any Borel measure* r *on* S,

$$\int_S \phi(s)\, dr(s) = \int_{\partial K} \phi\left(\frac{f}{\|f\|}\right) dn(f),$$

where n *is the Borel measure on* ∂K *defined by* $n(\cdot) = r(N(\cdot))$. *On the other hand,*

$$\int_{\partial K} \phi\left(\frac{f}{\|f\|}\right) dn(f) = \int_{\partial K} \phi(f) \frac{1}{\|f\|} dn(f)$$

Since K *is bounded and* $0 \in K^\circ$, *there exist* $\alpha, \beta > 0$ *such that* $\alpha < \|f\| < \beta$ *for all* $f \in \partial K$, *hence* $\frac{1}{\|f\|} \in L^1(n)^+$ *on* ∂K. *Define* $m(A) = \int_A \frac{1}{\|f\|} dn(f)$ *for all Borel subsets* A *of* ∂K *to obtain*

$$\int_S \phi(s)\, dr(s) = \int_{\partial K} \phi(f)\, dm(f).$$

A Borel measure m on a symmetric (about 0) Borel subset B of F is *even* if $m(A) = m(-A)$ for any Borel subset A of B (clearly ∂K is a symmetric compact subset of F).

Proposition 8 *If* K *is a (generalized) zonoid, then there exists an even (signed) measure* m *on* ∂K *such that*

$$I(x) = \int_{\partial K} |\langle x, \cdot\rangle|\, dm,$$

for all $x \in E$. *Such* m *is unique.*

Proof. By Bolker (1969), there exists an even (signed) measure r on S such that

$$I(v) = \max_{f \in K} \langle v, f\rangle = \max_{f \in K} (\widetilde{v}, f) = \int_S |(\widetilde{v}, s)|\, dr(s) = \int_S |\langle v, s\rangle|\, dr(s)$$

for all $v \in V$. By Petty (1961), r is unique. Notice that $|\langle v, \cdot\rangle|$ is continuous and positively homogeneous and apply Remark 2 (with n and m defined as there) to obtain $I(v) = \int_{\partial K} |\langle v, f\rangle|\, dm(f)$.

Let $x = v^\perp(x) + v(x) \in E$, then

$$I(x) = I(v(x)) = \int_{\partial K} |\langle v(x), f\rangle|\, dm(f) = \int_{\partial K} |\langle x, f\rangle|\, dm(f).$$

Next we show that m is even and unique.

For all Borel subsets A of ∂K, $N(-A) = \left\{\frac{-g}{\|-g\|} : g \in A\right\} = -N(A)$, $n(-A) = r(N(-A)) = r(-N(A)) = r(N(A)) = n(A)$, that is n is even.

On the other hand, $m(A) = \int_A \frac{1}{\|g\|} dn\,(g) = \int_{\partial K} 1_A\,(g) \frac{1}{\|g\|} dn\,(g)$; apply the change of variable $M : \partial K \to \partial K$ defined by $M\,(x) = -x$ to obtain $m(A) = \int_{\partial K} 1_A\,(-g) \frac{1}{\|-g\|} dnM\,(g)$. Since $nM = n$, then $m(A) = \int_{\partial K} 1_{-A}\,(g) \frac{1}{\|g\|} dn\,(g) = m\,(-A)$, and m is even.

Finally, let $\bar{m}$ be an even measure on ∂K such that $I(x) = \int_{\partial K} |\langle x, f\rangle|\, d\bar{m}\,(f)$ for all $x \in E$. Set $\bar{n}\,(A) = \int_A \|f\|\, d\bar{m}\,(f) = \int_{\partial K} 1_A\,(f)\, \|f\|\, d\bar{m}\,(f)$ for any Borel subsets A of ∂K. If $\widetilde{v} \in F$:

$$\begin{aligned}
\int_S |(\widetilde{v}, s)|\, dr\,(s) &= \int_S |\langle v, s\rangle|\, dr\,(s) = \int_{\partial K} |\langle v, f\rangle|\, d\bar{m}\,(f) = \\
&= \int_{\partial K} \left|\left\langle v, \frac{f}{\|f\|}\right\rangle\right| \|f\|\, d\bar{m}\,(f) = \\
&= \int_{\partial K} \left|\left\langle v, \frac{f}{\|f\|}\right\rangle\right| d\bar{n}\,(f) = \\
&= \int_S |\langle v, s\rangle|\, d\bar{n}N^{-1}\,(s) \\
&= \int_S |(\widetilde{v}, s)|\, d\bar{n}N^{-1}\,(s)\,.
\end{aligned}$$

Then, $r = \bar{n}N^{-1}$, whence $n = rN = \bar{n}N^{-1}N = \bar{n}$, and $m(A) = \int_{\partial K} 1_A\,(f) \frac{1}{\|f\|} dn\,(f) = \int_{\partial K} 1_A\,(f) \frac{1}{\|f\|} d\bar{n}\,(f) = \int_{\partial K} 1_A\,(f) \frac{1}{\|f\|} \|f\|\, d\bar{m}\,(f) = \bar{m}\,(A)$ for all Borel subsets A of ∂K. ■

Similarly to what done in Weil (1976), let $\mathcal{E}$ be the space of all (positively homogeneous) functions ϕ on F such that there exists a signed, even Borel measure m_ϕ on ∂K (or r_ϕ on S) such that

$$\phi\,(g) = \int_{\partial K} |(g, \cdot)|\, dm_\phi \text{ (or } \phi\,(g) = \int_S |(g, \cdot)|\, dr_\phi)$$

(m_ϕ and r_ϕ are unique), endowed with the topology of weak convergence of the generating signed measures. Consider the linear and continuous functional on $\mathcal{E}$ defined by

$$T\,(\phi) = m_\phi\left(\widetilde{I}\right),$$

For all $f \in \partial K$, consider the function $|(\cdot, f)|$. For all $g \in F$, $|(g, f)| = \frac{1}{2}|(g, f)| + \frac{1}{2}|(g, -f)| = \int_{\partial K} |(g, \cdot)|\, d\left(\frac{1}{2}\delta_f + \frac{1}{2}\delta_{-f}\right)$, hence, $|(\cdot, f)| \in \mathcal{E}$ and $m_{|(\cdot, f)|} = \frac{1}{2}\delta_f + \frac{1}{2}\delta_{-f}$. Finally $T\,(|(\cdot, f)|) = \int_{\partial K} \widetilde{I} d\left(\frac{1}{2}\delta_f + \frac{1}{2}\delta_{-f}\right) = \frac{1}{2}\widetilde{I}\,(f) + \frac{1}{2}\widetilde{I}\,(-f) = \widetilde{I}\,(f)$.

For all $h \in F$, there exists $\alpha \geq 0$ and $f \in \partial K$ such that $h = \alpha f$, then $\widetilde{I}\,(h) = \widetilde{I}\,(\alpha f) = \alpha\widetilde{I}\,(f) = \alpha T\,(|(\cdot, f)|) = T\,(\alpha\,|(\cdot, f)|) = T\,(|(\cdot, \alpha f)|) = T\,(|(h, \cdot)|)$. We can conclude that, for all $x = v^\perp\,(x) + v\,(x) \in E$,

$$\begin{aligned}
I(x) &= I\,(v\,(x)) = \widetilde{I}\left(\widetilde{v\,(x)}\right) \\
&= T\left(\left|\left(\widetilde{v\,(x)}, \cdot\right)\right|\right) = T\,(|\langle v\,(x)\,, \cdot\rangle|) \\
&= T\,(|\langle x, \cdot\rangle|)\,.
\end{aligned}$$

For further details and interpretation of T as a distribution see Weil (1976).

6.2 Compact sets of countably additive set functions

The vector space generated by all finitely additive probabilities is denoted by $ba(\Omega, \mathfrak{F})$ and its elements are denoted by $\mu : \mathfrak{F} \to \mathbb{R}$. Moreover, $ca(\Omega, \mathfrak{F})$ is the vector subspace of $ba(\Omega, \mathfrak{F})$ consisting of countably additive set functions.

As well known, $(ba(\Omega, \mathfrak{F}), \|\cdot\|_{\text{var}})$ is isometrically isomorphic to the norm dual of $\left(\mathcal{B}(\Omega, \mathfrak{F}), \|\cdot\|_{\sup}\right)$.[6] Therefore $ba(\Omega, \mathfrak{F})$ is endowed with a norm topology, a weak topology, a weak* topology, and the product topology inherited by $ba(\Omega, \mathfrak{F})$ as a subset of $\mathbb{R}^{\mathfrak{F}}$. Notice that the product topology coincides with the topology induced on $ba(\Omega, \mathfrak{F})$ by the subspace $\mathcal{B}_0(\Omega, \mathfrak{F})$ of its predual. Next lemma, which has an Eberlein-Smulian flavor, shows the equivalence of different notions of compactness for subsets of $ba(\Omega, \mathfrak{F})$. It is a simple variation on some results of Dunford and Schwartz (1958) Section IV.9. For a proof we refer to Chateauneuf, Maccheroni, Marinacci, and Tallon (2002).

Lemma 9 *Let $\mathbb{M}$ be a subset of $ba(\Omega, \mathfrak{F})$. Consider the following statements:*

a) $\mathbb{M}$ is weak compact.*

b) $\mathbb{M}$ is sequentially weak compact.*

c) $\mathbb{M}$ is cluster point weak compact.*

d) $\mathbb{M}$ is product compact.

e) $\mathbb{M}$ is weak compact.

We have that e) $\Leftrightarrow$ b) $\Rightarrow$ a) $\Leftrightarrow$ d) $\Rightarrow$ c), while all the conditions are equivalent if $\mathbb{M} \subseteq ca(\Omega, \mathfrak{F})$.

The following result provides conditions under which a set of positive finitely additive set functions is contained in $ca(\Omega, \mathfrak{F})$.

Lemma 10 *Let $\mathbb{M}$ be a weak* compact subset of $ba^+(\Omega, \mathfrak{F})$. The following statements are equivalent:*

1. *$\mathbb{M} \subseteq ca(\Omega, \mathfrak{F})$.*
2. *For all $X_n, X \in \mathcal{B}(\Omega, \mathfrak{F})$ such that $X_n \downarrow X$, $\max_{\mu \in \mathbb{M}} \int_\Omega X_n d\mu \to \max_{\mu \in \mathbb{M}} \int_\Omega X d\mu$.*
3. *For all $A_n \in \mathfrak{F}$ such that $A_n \downarrow \emptyset$, $\max_{\mu \in \mathbb{M}} \mu(A_n) \to 0$.*

---

[6] For all $\mu \in ba(\Omega, \mathfrak{F})$, $\|\mu\|_{\text{var}} = \sup \sum_{i=1}^{N} |\mu(A_i)|$, where the sup is taken over all finite partitions of Ω in $\mathfrak{F}$. For all $X \in \mathcal{B}(\Omega, \mathfrak{F})$, $\|X\|_{\sup} = \sup |X(\omega)|$, where the sup is taken over all $\omega \in \Omega$.

Proof. 1.⇒2. For all $n \geq 1$ consider the function

$$\begin{array}{rccc} \langle X_n, \cdot \rangle : & \mathbb{M} & \to & \mathbb{R} \\ & \mu & \mapsto & \int_\Omega X_n d\mu \end{array}$$

- $\langle X_n, \cdot \rangle$ is weak* continuous on $\mathbb{M}$ for all $n \geq 1$.
- For all $\mu \in \mathbb{M}$, $\langle X_n, \mu \rangle \mapsto \langle X, \mu \rangle$ (monotone convergence theorem) and, moreover $\langle X, \cdot \rangle$ is weak* continuous on $\mathbb{M}$.
- For all $n \geq 1$ and all $\mu \in \mathbb{M}$, $\langle X_n, \mu \rangle \geq \langle X_{n+1}, \mu \rangle$.

Therefore, by the Dini Theorem $\langle X_n, \cdot \rangle$ uniformly converges to $\langle X, \cdot \rangle$. A fortiori, $\max_{\mu \in \mathbb{M}} \langle X_n, \mu \rangle \to \max_{\mu \in \mathbb{M}} \langle X, \mu \rangle$.

2.⇒3. Obvious.

3.⇒1. For all $A_n \in \mathfrak{F}$ such that $A_n \downarrow \emptyset$, and all $\bar{\mu} \in \mathbb{M}$: $\bar{\mu}(A_n) \leq \max_{\mu \in \mathbb{M}} \mu(A_n) \to 0$. ■

6.3 Suprema of integrals

The final lemma is an approximation result.

Lemma 11 *Let $\mathbb{P} \subseteq ba(\Omega, \mathfrak{F})$ be a set of probabilities on $\mathfrak{F}$ and $X \in \mathcal{M}(\Omega, \mathfrak{F})$. We have:*

$$\sup_{P \in \mathbb{P}} \int_\Omega X dP = \sup_{P \in \mathbb{P}} \lim_n \int_\Omega (X \wedge n)\, dP = \lim_n \sup_{P \in \mathbb{P}} \int_\Omega (X \wedge n)\, dP.$$

Proof. It is easy to prove that

$$\lim_n \int_\Omega (X \wedge n)\, dP = \int_\Omega X dP, \tag{8}$$

hence

$$\sup_{P \in \mathbb{P}} \lim_n \int_\Omega (X \wedge n)\, dP = \sup_{P \in \mathbb{P}} \int_\Omega X dP.$$

But for any $P \in \mathbb{P}$ and any $n \in \mathbb{N}$

$$\int_\Omega X dP \geq \int_\Omega (X \wedge n)\, dP,$$

and for any $n \in \mathbb{N}$

$$\sup_{P \in \mathbb{P}} \int_\Omega X dP \geq \sup_{P \in \mathbb{P}} \int_\Omega (X \wedge n)\, dP,$$

but $\sup_{P \in \mathbb{P}} \int_\Omega (X \wedge n)\, dP$ is an increasing sequence, then

$$\sup_{P \in \mathbb{P}} \int_\Omega X dP \geq \lim_n \sup_{P \in \mathbb{P}} \int_\Omega (X \wedge n)\, dP.$$

Suppose that $\sup_{P\in\mathbb{P}} \int_\Omega XdP > \lim_n \sup_{P\in\mathbb{P}} \int_\Omega (X\wedge n)\, dP$. Then there exists $P^* \in \mathbb{P}$ such that

$$\int_\Omega XdP^* > \lim_n \sup_{P\in\mathbb{P}} \int_\Omega (X\wedge n)\, dP \geq \lim_n \int_\Omega (X\wedge n)\, dP^* = \int_\Omega XdP^*,$$

a contradiction. ■

6.4 Proofs of the results

Proof of Theorem 1. Assume first that S.1 is satisfied. Set $\mathcal{Z} = \langle\mathcal{Y}\rangle + \mathcal{B}_0(\Omega,\mathfrak{F}) \subseteq \mathcal{B}(\Omega,\mathfrak{F})$. Define a functional $\widehat{H}: \mathcal{Z} \to \mathbb{R}$ by

$$\widehat{H}(Z) = H(Z+\gamma) - \gamma$$

if $\gamma \in \mathbb{R}^+$ is such that $Z+\gamma \in \langle\mathcal{Y}\rangle^+ + \mathcal{B}_0^+(\Omega,\mathfrak{F}) \subseteq \mathcal{X}$ (for convenience, throughout the proof we write γ in place of $\gamma 1_\Omega$).

$\widehat{H}$ is well defined. For, let $Z = X_0 + Y$ with $X_0 \in \mathcal{B}_0(\Omega,\mathfrak{F})$ and $Y \in \langle\mathcal{Y}\rangle$. Clearly there exist $\xi_0, \theta \in \mathbb{R}^+$ such that $X_0 + \xi_0 \in \mathcal{B}_0^+(\Omega,\mathfrak{F})$ and $Y+\theta \in \langle\mathcal{Y}\rangle^+$. By setting $\gamma = \xi_0 + \theta$, we obtain $Z+\gamma \in \langle\mathcal{Y}\rangle^+ + \mathcal{B}_0^+(\Omega,\mathfrak{F})$. Hence, $\widehat{H}$ is defined on the whole $\mathcal{Z}$. Next we show that its definition does not depend on γ. Assume $Z+\gamma, Z+\beta \in \langle\mathcal{Y}\rangle^+ + \mathcal{B}_0^+(\Omega,\mathfrak{F})$. W.l.o.g. $\gamma > \beta$ and

$$H(Z+\gamma) - \gamma = H(Z+\beta+\gamma-\beta) - \gamma.$$

On the other hand, $\frac{Z+\beta}{\gamma-\beta} \in \langle\mathcal{Y}\rangle^+ + \mathcal{B}_0^+(\Omega,\mathfrak{F})$, $\frac{\gamma-\beta}{\gamma-\beta} = 1 \in \mathcal{Y}$, and $\frac{Z+\beta}{\gamma-\beta} + \frac{\gamma-\beta}{\gamma-\beta} = \frac{Z+\gamma}{\gamma-\beta} \in \langle\mathcal{Y}\rangle^+ + \mathcal{B}_0^+(\Omega,\mathfrak{F})$. Therefore, by A.2 and A.4

$$\begin{aligned} H(Z+\beta+\gamma-\beta) - \gamma &= H\left((\gamma-\beta)\left(\frac{Z+\beta}{\gamma-\beta} + \frac{\gamma-\beta}{\gamma-\beta}\right)\right) - \gamma \\ &= (\gamma-\beta)\left(H\left(\frac{Z+\beta}{\gamma-\beta}\right) + 1\right) - \gamma \\ &= (\gamma-\beta) H\left(\frac{Z+\beta}{\gamma-\beta}\right) + \gamma - \beta - \gamma \\ &= H(Z+\beta) - \beta. \end{aligned}$$

We conclude that $H(Z+\gamma) - \gamma = H(Z+\beta) - \beta$.

Clearly if $Z \in \langle\mathcal{Y}\rangle^+ + \mathcal{B}_0^+(\Omega,\mathfrak{F})$, then $\widehat{H}(Z) = H(Z+0) - 0 = H(Z)$; in particular, $\widehat{H}(1) = 1$ and $\widehat{H}(0) = 0$.

$\widehat{H}$ is positively homogeneous. In fact, if $\gamma \in \mathbb{R}^+$ is such that $Z+\gamma \in \langle\mathcal{Y}\rangle^+ + \mathcal{B}_0^+(\Omega,\mathfrak{F})$ and $\alpha \geq 0$, then $\alpha Z + \alpha\gamma \in \langle\mathcal{Y}\rangle^+ + \mathcal{B}_0^+(\Omega,\mathfrak{F})$, by A.2

$$\begin{aligned} \widehat{H}(\alpha Z) &= H(\alpha Z + \alpha\gamma) - \alpha\gamma = H(\alpha(Z+\gamma)) - \alpha\gamma \\ &= \alpha H(Z+\gamma) - \alpha\gamma = \alpha\widehat{H}(Z). \end{aligned}$$

$\widehat{H}$ is subadditive. For, let $\gamma, \gamma' \in \mathbb{R}^+$ be such that $Z+\gamma, Z+\gamma' \in \langle \mathcal{Y} \rangle^+ + \mathcal{B}_0^+ (\Omega, \mathfrak{F})$, then $Z+\gamma+Z'+\gamma' \in \langle \mathcal{Y} \rangle^+ + \mathcal{B}_0^+ (\Omega, \mathfrak{F})$. By A.3

$$\begin{aligned} \widehat{H}(Z+Z') &= H(Z+Z'+(\gamma+\gamma')) - (\gamma+\gamma') = \\ &= H((Z+\gamma)+(Z'+\gamma')) - (\gamma+\gamma') \leq \\ &\leq H(Z+\gamma) + H(Z'+\gamma') - \gamma - \gamma' = \\ &= \widehat{H}(Z) + \widehat{H}(Z'). \end{aligned}$$

This proves subadditivity. Now assume $Z \in \mathcal{Z}$ and $Y \in \mathcal{Y}$, there exists $n \in \mathbb{N}$ such that $Y + n \in \mathcal{Y}^+$. Let $\gamma \in \mathbb{R}^+$ be such that $Z + \gamma \in \langle \mathcal{Y} \rangle^+ + \mathcal{B}_0^+ (\Omega, \mathfrak{F})$. A.4 and an argument analogous to the one just used yield $\mathcal{Y}$-additivity.

$\widehat{H}(Z) \leq 0$ if $Z \leq 0$. Let $Z \leq 0$ and $\gamma \in \mathbb{R}^+$ such that $Z + \gamma \in \langle \mathcal{Y} \rangle^+ + \mathcal{B}_0^+ (\Omega, \mathfrak{F})$:

$$\widehat{H}(Z) = H(Z+\gamma) - \gamma \leq \sup(Z+\gamma) - \gamma = \sup Z \leq 0.$$

In sum, $\widehat{H} : \mathcal{Z} \to \mathbb{R}$ is a sublinear, $\mathcal{Y}$-additive functional such that $\widehat{H}(Z) \leq 0$ if $Z \leq 0$. By Lemma 6, there exists a convex and $\sigma(\mathcal{Z}', \mathcal{Z})$-compact set Φ of linear and monotone functionals on $\mathcal{Z}$, with $\phi(Y) = \widehat{H}(Y) = V(Y)$ for all $Y \in \mathcal{Y}$ and all $\phi \in \Phi$, such that

$$\widehat{H}(Z) = \max_{\phi \in \Phi} \phi(Z)$$

for all $Z \in \mathcal{Z}$. Since $\phi(1) = \widehat{H}(1) = 1$, for each ϕ there exists a unique (since $\mathcal{B}_0(\Omega, \mathfrak{F}) \subseteq \mathcal{Z}$) probability P_ϕ on $\mathfrak{F}$ such that

$$\phi(Z) = \int_\Omega Z dP_\phi$$

for all $Z \in \mathcal{Z}$. Thus

$$\widehat{H}(Z) = \max_{\phi \in \Phi} \int_\Omega Z dP_\phi$$

for all $Z \in \mathcal{Z}$, and

$$V(Y) = \phi(Y) = \int_\Omega Y dP_\phi$$

for all $Y \in \mathcal{Y}$ and all $\phi \in \Phi$. Let $\mathbb{Q} = \{P_\phi\}_{\phi \in \Phi}$. It is easy to show that $\mathbb{Q}$ is convex and weak* compact. By outer continuity, if $\{A_n\} \subseteq \mathfrak{F}$ and $A_n \downarrow \emptyset$, then $Q(A_n) = \int_\Omega 1_{A_n} dQ \leq H(1_{A_n}) \to 0$ for all $Q \in \mathbb{Q}$. Hence the elements of $\mathbb{Q}$ are countably additive.

Let $X \in \mathcal{X}$ be bounded and measurable. There exists a sequence $\{X_n\}_{n \geq 1} \subseteq \mathcal{B}_0(\Omega, \mathfrak{F})^+$ that decreases to X. By A.6 and Lemma 10,

$$H(X) = \lim_n H(X_n) = \lim_n \left(\max_{Q \in \mathbb{Q}} \int_\Omega X_n dQ \right) = \max_{Q \in \mathbb{Q}} \int_\Omega X dQ. \tag{9}$$

Suppose that X is not necessarily bounded. Since $X \wedge n$ is bounded, by (9) and Lemma 11, we have

$$\begin{aligned} H(X) &= \lim_n H(X \wedge n) = \lim_n \sup_{Q \in \mathbb{Q}} \int_\Omega (X \wedge n)\, dQ \\ &= \sup_{Q \in \mathbb{Q}} \lim_n \int_\Omega (X \wedge n)\, dQ = \sup_{Q \in \mathbb{Q}} \int_\Omega X dQ. \end{aligned}$$

Uniqueness of $\mathbb{Q}$ is proved like uniqueness of Φ in Lemma 6. This completes the proof of 1. $\Rightarrow$ 2. if S.1 is satisfied.

Conversely, consider a functional defined on $\mathcal{X}$ by

$$H(X) = \sup_{Q \in \mathbb{Q}} \int_\Omega X dQ$$

where $\mathbb{Q}$ is a weak* compact and convex set of countably additive risk neutral probabilities. It obviously satisfies A.1-A.4.

For all $X \in \mathcal{X}$, by Lemma 11,

$$\begin{aligned} \lim_n H(X \wedge n) &= \lim_n \sup_{Q \in \mathbb{Q}} \int_\Omega (X \wedge n)\, dQ = \sup_{Q \in \mathbb{Q}} \lim_n \int_\Omega (X \wedge n)\, dQ \\ &= \sup_{Q \in \mathbb{Q}} \int_\Omega X dQ = H(X). \end{aligned}$$

and H satisfies A.5. If $\{X_n\}_{n \geq 1} \subseteq \mathcal{B}_0^+(\Omega, \mathfrak{F})$ is such that $X_n \downarrow X \in \mathcal{X}$, for all $n \geq 1$, by Lemma 10,

$$\lim_n H(X_n) = \lim_n \left(\max_{Q \in \mathbb{Q}} \int_\Omega X_n dQ \right) = \max_{Q \in \mathbb{Q}} \int_\Omega X dQ = H(X),$$

and A.6 is satisfied. A similar argument holds for S.2. ■

Proof of Theorem 3. The proof only assumes that $\mathbb{Q} \subseteq ba(\Omega, \mathfrak{F})$ in order to apply the same argument to prove Proposition 5. Apply Theorem 1 to obtain

$$H(X) = \sup_{Q \in \mathbb{Q}} \int_\Omega X dQ$$

for all $X \in \mathcal{X}$. Notice that A.7 guarantees that $H(X) - H(-X)$ is linear on $\mathcal{B}_0(\Omega, \mathfrak{F})$, hence, by Lemma 7, $\mathbb{Q}$ is symmetric. Let P be the center of symmetry of $\mathbb{Q}$. Again by Lemma 7

$$\int_\Omega X dP = \frac{1}{2}(H(X) - H(-X)) = \frac{1}{2} \max_{Q \in \mathbb{Q}} \int_\Omega X dQ + \frac{1}{2} \min_{R \in \mathbb{Q}} \int_\Omega X dR$$

for all $X \in \mathcal{B}_0(\Omega, \mathfrak{F})$. Therefore, for all $A \in \mathfrak{F}$

$$P(A) = \int_\Omega 1_A dP = \frac{1}{2} \max_{Q \in \mathbb{Q}} \int_\Omega 1_A dQ + \frac{1}{2} \min_{R \in \mathbb{Q}} \int_\Omega 1_A dR = \frac{1}{2} \max_{Q \in \mathbb{Q}} Q(A) + \frac{1}{2} \min_{R \in \mathbb{Q}} R(A).$$

Let $X \in \mathcal{X}$. Clearly, $H(X) < \infty$ implies $\int_\Omega X dP < \infty$. Conversely, if $\int_\Omega X dP < \infty$, since $Q(A) \leq 2P(A)$ for all $Q \in \mathbb{Q}$ and all $A \in \mathfrak{F}$, then

$$H(X) = \sup_{Q\in\mathbb{Q}} \int_\Omega X dQ \leq 2\int_\Omega X dP - \inf X < \infty.$$

If $H(X) < \infty$,

$$H(X) = \sup_{Q\in\mathbb{Q}} \int_\Omega X dQ = \int_\Omega X dP + \sup_{Q\in\mathbb{Q}} \left(\int_\Omega X dQ - \int_\Omega X dP\right).$$

Since $\mathbb{Q}$ is symmetric

$$\sup_{Q\in\mathbb{Q}} \left(\int_\Omega X dQ - \int_\Omega X dP\right) = \sup_{Q\in\mathbb{Q}} \left(\int_\Omega X dP - \int_\Omega X dQ\right).$$

Next we show that $\sup_{Q\in\mathbb{Q}} \left(\int_\Omega X dQ - \int_\Omega X dP\right) = \frac{1}{2} \sup_{S,R\in\mathbb{Q}} \left(\int_\Omega X dS - \int_\Omega X dR\right)$. For all $Q \in \mathbb{Q}$,

$$2\left(\int_\Omega X dP - \int_\Omega X dQ\right) = \int_\Omega X d(2P - Q) - \int_\Omega X dQ \leq \sup_{S,R\in\mathbb{Q}} \left(\int_\Omega X dS - \int_\Omega X dR\right),$$

and hence $\sup_{Q\in\mathbb{Q}} \left(\int_\Omega X dQ - \int_\Omega X dP\right) \leq \frac{1}{2} \sup_{S,R\in\mathbb{Q}} \left(\int_\Omega X dS - \int_\Omega X dR\right)$. Conversely, if $\sup_{Q\in\mathbb{Q}} \left(\int_\Omega X dQ - \int_\Omega X dP\right) < \frac{1}{2} \sup_{S,R\in\mathbb{Q}} \left(\int_\Omega X dS - \int_\Omega X dR\right)$, there exist R and S such that

$$\begin{aligned}
\int_\Omega X dS - \int_\Omega X dR &> 2\sup_{Q\in\mathbb{Q}} \left(\int_\Omega X dQ - \int_\Omega X dP\right) = \\
&= \sup_{Q\in\mathbb{Q}} \left(\int_\Omega X dQ - \int_\Omega X dP\right) + \sup_{Q\in\mathbb{Q}} \left(\int_\Omega X dP - \int_\Omega X dQ\right) \geq \\
&\geq \int_\Omega X dS - \int_\Omega X dP + \int_\Omega X dP - \int_\Omega X dR,
\end{aligned}$$

a contradiction.

If $H(X) = \infty$,

$$H(X) = \int_\Omega X dP = \int_\Omega X dP + \frac{1}{2} \sup_{S,R\in\mathbb{Q}} \left|\int_\Omega X dS - \int_\Omega X dR\right|.$$

The rest is trivial. ■

Proof of Theorem 4. Just observe that for all $X \in \mathcal{B}(\Omega, \mathfrak{F})$, $\mathrm{Amb}_{\mathbb{Q}}(X) = \max_{\mu\in\mathbb{Q}-\mathbb{Q}} \int X d\mu$ and that, if $\mathbb{Q}$ is a (generalized) zonoid with center of symmetry P, then $\mathbb{Q} - \mathbb{Q} = 2(\mathbb{Q} - P)$ is a (generalized) zonoid centered in 0; finally, apply Proposition 8. ■

Proof of Proposition 5. For the first part of the proposition, it suffices to follow the path of the proof of Theorem 1 with $\mathcal{B}(\Omega, \mathfrak{F})^+$ in place of $\langle \mathcal{Y} \rangle^+ + \mathcal{B}_0(\Omega, \mathfrak{F})^+$. Analogously the second part is proved like Theorem 3, with the exception of weak compactness of $\mathbb{Q}$. Notice for all $Q \in \mathbb{Q}$ and $A \in \mathfrak{F}$, $0 \leq Q(A) \leq 2P(A)$ for all $A \in \mathfrak{F}$. For all $\varepsilon > 0$, if $P(A) < \frac{\varepsilon}{2}$, then $Q(A) < \varepsilon$, that is

$$\lim_{P(A) \to 0} Q(A) = 0$$

uniformly for $Q \in \mathbb{Q}$. By Theorem IV.9.12 in Dunford and Schwartz (1958), $\mathbb{Q}$ is relatively sequentially weak compact, and, by the Eberlein-Smulian Theorem, $\mathbb{Q}$ is relatively weak compact. But $\mathbb{Q}$ is weak* closed, then it is weak closed, and compact. ■

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