Ranking Intersecting Lorenz Curves

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Abstract

This paper is concerned with the problem of ranking Lorenz curves in situations where the Lorenz curves intersect and no unambiguous ranking can be attained without introducing weaker ranking criteria than first-degree Lorenz dominance. To deal with such situations two alternative sequences of nested dominance criteria between Lorenz curves are introduced. At the limit the systems of dominance criteria appear to depend solely on the income share of either the worst-off or the best-off income recipient. This result suggests two alternative strategies for increasing the number of Lorenz curves that can be strictly ordered; one that focuses on changes that take place in the lower part of the income distribution and the other that focuses on changes that concern the upper part of the income distribution. Furthermore, it is demonstrated that the sequences of dominance criteria characterize two separate systems of nested subfamilies of inequality measures and thus provide a method for identifying the least restrictive social preferences required to reach an unambiguous ranking of Lorenz curves.

Keywords: The Lorenz curve, partial orderings, rank-dependent measures of inequality, generalized Gini families of inequality measures, principles of transfers and mean-spread-preserving transformations.

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1. Introduction

In empirical analyses of income distribution it is common practice to make separate comparisons of mean incomes and Lorenz curves. The Lorenz curve, which was introduced by Lorenz (1905) as a representation of inequality, is concerned with income shares without taking account of differences in mean incomes. Thus, adopting the Lorenz curve as a basis for judging between income distributions means that we focus solely on distributional aspects. The widespread use of the Lorenz curve in applied work shows that focusing on distributional aspects is of interest in its own right[1], irrespective of how we judge between level of mean income and degree of inequality in cases where they conflict. For welfare judgments about the trade-off between mean income and inequality we refer to Shorrocks (1983), Ebert (1987) and Lambert (1985, 1993a). Following Ebert's approach we note, however, that orderings defined on Lorenz curves can also be used as a basis for deriving social welfare orderings and related welfare functions. As emphasized by Ebert this procedure of deriving social welfare functions is of particular interest since it, in contrast to the conventional approach, explicitly takes into account value judgments about the trade-off between mean income and income inequality.

Ranking Lorenz curves in accordance with first-degree Lorenz dominance means that the higher of non-intersecting Lorenz curves is preferred. However, since Lorenz curves may intersect, which is often the case in applied economics, other ranking criteria than first-degree Lorenz dominance are needed to reach an unambiguous conclusion. The standard practice for ranking intersecting Lorenz curves is to apply summary measures of inequality, such as the Gini coefficient. However, as it may be difficult to find a single measure that gains a wide degree of support, it is of interest to search for alternative ranking criteria that are stronger than single measures of inequality and weaker than first-degree Lorenz dominance. Restricting attention to distributions of equal means, Kolm (1969) and Atkinson (1970) observed that dominance of non-intersecting Lorenz curves and second-degree stochastic dominance are identical requirements, and thus recognized that the family of inequality measures derived from utilitarian social welfare functions with concave utility functions yields a characterization of the criterion of non-intersecting Lorenz curves. This result suggests the hypothesis that second-degree Lorenz dominance imposes the restriction of positive third derivative on the utility function of the utilitarian inequality measures, where second-degree Lorenz dominance is defined analogous to second-degree stochastic dominance. Unfortunately, this hypothesis has to be rejected since second-degree Lorenz dominance and third-degree stochastic dominance do not coincide. However, useful analyses of the implications of third-degree stochastic dominance on measurement of inequality and social welfare have been provided by Shorrocks and Foster (1987), Dardanoni and Lambert (1988) and Davis and Hoy (1994, 1995), whilst Muliere and Scarcini (1989) and Zoli (1999)

[1] See e.g. Coder et al. (1989) and Atkinson et al. (1995) who make cross-country comparisons of Lorenz curves allowing for differences between countries in level of income.

have examined the implications of applying second-degree Lorenz dominance as a criterion for ranking Lorenz curves. While the majority of the results in these papers concern the case of singly intersecting Lorenz curves the latter four papers provide results for the case of multiple crossings as well.[2] The results of Muliere and Scarcini (1989) and Zoli (1999) suggest that there may be a closer relationship between Lorenz dominance (of various degrees) and rank-dependent measures of inequality than between Lorenz dominance and utilitarian measures of inequality. This paper pursues this idea by exploring the relationship between various Lorenz dominance criteria and the family of rank-dependent inequality measures introduced by Mehran (1976).

As will be demonstrated in Sections 2 and 3, second-degree Lorenz dominance forms a natural basis for the construction of two separate hierarchical sequences of partial orderings (dominance criteria), where one sequence focuses on the lower part of the Lorenz curve while the other focuses on the upper part of the Lorenz curve. The hierarchical and nested structure of the dominance criteria appears to be useful in empirical applications since we are allowed to identify the lowest degree of dominance required to reach unambiguous rankings of Lorenz curves. Moreover, Section 3 demonstrates that the two hierarchical sequences of Lorenz dominance criteria divide the Mehran family of inequality measures into two corresponding hierarchical systems of nested subfamilies. Section 4 uses these results to arrange the members of two different generalized Gini families of inequality measures into subfamilies according to their relationship to Lorenz dominance of various degrees. Actually, each of the subfamilies proves to give a complete characterization of the corresponding degree of Lorenz dominance. Section 5 briefly summarizes the conclusions of the paper and discusses the use of the obtained results as a basis for deriving dominance criteria for generalized Lorenz curves.

2. Lorenz dominance of first and second degree

The Lorenz curve L for a cumulative distribution F with mean μ is defined by

$$L(u)=\frac{1}{\mu}\int_0^u F^{-1}(t)\,dt,\ \ 0\le u\le 1, \tag{1}$$

where F^{-1} is the left inverse of F.

Under the restriction of equal mean incomes the problem of ranking Lorenz curves formally corresponds to the problem of choosing between uncertain prospects. This relationship has been utilized by e.g. Kolm (1969) and Atkinson (1970) to characterize the criterion of non-intersecting Lorenz curves in cases of equal mean incomes. Atkinson reinterpreted the standard theory of choice

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[2] See Lambert (1993b) for a discussion of applying Lorenz dominance criteria as basis for evaluating distributional effects of tax reforms.

under uncertainty and demonstrated that inequality aversion can in fact be viewed as being equivalent to risk aversion. This was motivated by the fact that in cases of equal mean incomes the criterion of non-intersecting Lorenz curves is equivalent to second-degree stochastic dominance, which means that the Pigou-Dalton transfer principle is identical to the principle of mean preserving spread introduced by Rothschild and Stiglitz (1970). To perform inequality comparisons with Lorenz curves we can deal with distributions of relative incomes or alternatively simply abandon the assumption of equal means.[3] The latter approach normally forms the basis of empirical studies and will also be employed in this paper.

The criterion of non-intersecting Lorenz curves, called first-degree Lorenz dominance, is based on the following definition[4].

DEFINITION 1. A Lorenz curve L_1 is said to first-degree dominate a Lorenz curve L_2 if

$$L_1(u) \geq L_2(u) \text{ for all } u \in [0,1]$$

and the inequality holds strictly for some $u \in \langle 0,1 \rangle$.

A person who prefer the dominating one of non-intersecting Lorenz curves favors transfers of incomes which reduce the differences between the income shares of the donors and the recipients, and is therefore said to be inequality averse.

As noted above Kolm (1969) and Atkinson (1970) provided a welfare characterization of first-degree Lorenz dominance when the judgment about inequality is restricted to distributions with equal means. Moreover, the definition of the Atkinson family of inequality measures is motivated by this characterization result even though measures of the Atkinson family are proved to solely preserve first-degree Lorenz dominance. Thus, there may occur situations where Lorenz curves intersect and still all the members of the Atkinson family agree with respect to the ranking of the Lorenz curves. However, by dealing with income divided by the mean rather than income itself the Kolm-Atkinson characterization result also applies to cases of variable mean income: Let X_1 and X_2 be income variables with Lorenz curves L_1 and L_2 and means μ_1 and μ_2. Then the following statements are equivalent,

(i) $$L_1(u) \geq L_2(u) \text{ for all } u \in [0,1]$$

[3] The importance of focusing on relative incomes was acknowledged already by Plato who proposed that the ratio of the top income to the bottom should be less than four to one (see Cowell, 1977). See also Sen's (1992) discussion of relative deprivation and Smith's (1776) discussion of necessities.

[4] Note that most analyses of Lorenz dominance apply a definition that excludes the requirement of strict inequality for some u.

and the inequality holds strictly for some $u \in \langle 0,1 \rangle$,

(ii)
$$E\,U\left(\frac{X_1}{\mu_1}\right) \geq E\,U\left(\frac{X_2}{\mu_2}\right)$$

for all increasing concave utility functions U.

Accordingly, first-degree Lorenz dominance characterizes the following family of inequality measures,

(2)
$$I = 1 - U^{-1}\left(E\,U\left(\frac{X}{\mu}\right)\right),$$

where X is an income variable with mean μ and U^{-1} is the left inverse of U. Note that the Atkinson family of inequality measures is a subfamily of the family I defined by (2). In contrast to the inequality measures of the Atkinson family the remaining inequality measures of the utilitarian family (2) can not be expressed explicitly in terms of the equally distributed equivalent income but rather in terms of the equally distributed equivalent mean-normalized income. Therefore, the equally distributed equivalent mean-normalized income coincides with the equally distributed equivalent income divided by the mean solely for utility functions that define the Atkinson measures of inequality.

As explained in Section 1 the Mehran family of rank-dependent measures of inequality J_P appears to form a more convenient basis for judging the normative significance of various Lorenz dominance criteria than the utilitarian family (2)[5]. The J_P-family is defined by

(3)
$$J_P(L) = 1 - \int_0^1 P'(u)\,d\,L(u)$$

where L is the Lorenz curve and the weight-function P' is the derivative of a continuous, differentiable and concave distribution function defined on the unit interval. Note that P can be interpreted as a preference function that characterizes the inequality aversion profile of a person who employs J_P to judge between Lorenz curves. To ensure that J_P has the unit interval as its range the condition $P'(1) = 0$ is imposed on P.

For the sake of completeness the characterization results of first-degree Lorenz dominance given by Yaari (1988) is reproduced in Theorem 1 below, where **L** denotes the family of Lorenz

[5] Mehran (1976) introduced the J_P-family by relying on descriptive arguments. For alternative normative motivations of the J_P-family and various subfamilies of the J_P-family we refer to Donaldson and Weymark (1980, 1983), Weymark (1981), Yaari (1987,1988), Ben Porath and Gilboa (1994) and Aaberge(2000b).

curves and $\mathbf{P}_1$ is a class of preference functions defined by

$$\mathbf{P}_1 = \left\{ P : P' \text{ and } P'' \text{ are continuous on } [0,1],\ P'(t) > 0 \text{ and } P''(t) < 0 \text{ for } t \in \langle 0,1 \rangle, \text{ and } P'(1) = 0 \right\}.$$

THEOREM 1. (Yaari, 1988). *Let L_1 and L_2 be members of $\boldsymbol{L}$. Then the following statements are equivalent,*

(i) $L_1(u) \geq L_2(u)$ *for all* $u \in [0,1]$

and the inequality holds strictly for some $u \in \langle 0,1 \rangle$

(ii) $J_P(L_1) < J_P(L_2)$ *for all* $P \in \boldsymbol{P}_1$.

(Proof in Appendix.)

The characterization of the condition of first-degree Lorenz dominance provided by Theorem 1 shows that non-intersecting Lorenz curves can be ordered without specifying further the functional form of the preference function P other than P being strictly concave. This means that J_P satisfies the principle of transfers for concave P-functions.

To deal with situations where Lorenz curves intersect, a weaker ranking principle than first-degree Lorenz dominance is called for. In applied economics the standard approach in these cases is to apply summary measures of inequality, such as the Gini coefficient. The advantage of using summary measures is that they always yield a complete ranking of Lorenz curves and, moreover, provide a quantification of the extent of inequality. However, as there may be difficult to find a single measure that gain a wide degree of support, it is important to search for alternative ranking criteria that are stronger than single measures of inequality and weaker than first-degree Lorenz dominance.[6]

As noted above first-degree Lorenz dominance and second-degree stochastic dominance are formally identical ranking criteria for distributions with equal means. This suggests that we may draw on results from the literature on stochastic dominance when searching for weaker ranking conditions than first-degree Lorenz dominance[7]. As recognized by Muliere and Scarsini (1989) there is, however, no simple relationship between third-degree stochastic dominance and second-degree Lorenz dominance. Thus, a general characterization of second-degree Lorenz dominance or third-degree inverse stochastic dominance in terms of ordering conditions for the utilitarian measures of inequality defined by (2) is not obtained. However, by restricting to the case of singly intersecting Lorenz curves Shorrocks and Foster (1987) demonstrated that (2) (with $U''(x) < 0$ and $U'''(x) > 0$) is equivalent to

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[6] For the purpose of empirical applications Aaberge (2000a) provides arguments for applying a few specific summary measures of inequality, one of which is the Gini coefficient.

[7] In the context of expected utility theory Whitmore (1970) proposed a definition of third-degree stochastic dominance which implied certain restrictions on the first, second and third moments of the distributions. A definition of k^{th} degree stochastic dominance, restricted to distributions where the k first moments agree, was introduced by Chew (1983).

second-degree Lorenz dominance provided the Lorenz curve of the income distribution with the lower coefficient of variation crosses the other one from above. Dardanoni and Lambert (1988) have established similar characterization results for generalized Lorenz curves. Davis and Hoy (1995) provide a more general result that covers multiply intersecting Lorenz curves, but their result requires computation and comparison of the coefficients of variation for each of the actual intersections between the Lorenz curves.

Observe, however, that the characterization of second-degree stochastic dominance given by Hadar and Russel (1969) also applies to Lorenz curves in cases of unequal mean incomes. This suggests the following definition of second-degree upward Lorenz dominance.

DEFINITION 2. *A Lorenz curve L_1 is said to second-degree upward dominate a Lorenz curve L_2 if*

$$\int_0^u L_1(t)\,dt \geq \int_0^u L_2(t)\,dt \quad \text{for all } u \in [0,1]$$

and the inequality holds strictly for some u.

Since second-degree upward Lorenz dominance is not consistent with third-degree stochastic dominance it is neither consistent with Kolm's (1976) principle of diminishing transfers[8] which places more weight on a transfer between persons with a given income difference if these incomes are lower than if they are higher. To account for differences in the proportion of individuals between receivers and donors of income transfers Mehran (1976) introduced an alternative principle of diminishing transfers (denoted the principle of positional transfer sensitivity by Zoli, 1999), which for a fixed difference in ranks requires a given transfer of money from a richer to a poorer person to be more equalizing the lower it occurs in the income distribution. Thus, the essential difference between these two principles is that the transfers are made under the condition of a fixed income gap in the former case and a fixed difference in ranks in the latter case. Mehran (1976) demonstrated that J_P defined by (3) satisfies the principle of positional transfer sensitivity if and only if $P'''(t) > 0$[9]. However, as stated in Theorem 2 below the principles of transfers and positional transfer sensitivity (jointly) prove to be equivalent to the condition of second-degree upward Lorenz dominance.

The motivation for the notion "upward dominance" in Definition 2 relates to the fact that the aggregation starts from the lower part of the Lorenz curve. By contrast, an alternative ranking

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[8] Denoted aversion to downside inequality by Davis and Hoy (1994, 1995).

[9] Aaberge (2000a) demonstrated that J_P defined by (3) satisfies Kolm's principle of diminishing transfers under conditions which depend on the shape of the income distribution F as well as on the shape of the preference function P .

criterion to second-degree upward Lorenz dominance can be obtained by aggregating the Lorenz curve from above.

DEFINITION 3. *A Lorenz curve L_1 is said to second-degree downward dominate a Lorenz curve L_2 if*

$$\int_u^1 \left(1 - L_2(t)\right) dt \geq \int_u^1 \left(1 - L_1(t)\right) dt \quad \text{for all} \quad u \in [0,1]$$

and the inequality holds strictly for some u.

Note that second-degree downward as well upward Lorenz dominance preserves first-degree Lorenz dominance. Consequently, both dominance criteria are consistent with the Pigou-Dalton principle of transfers. The choice between second-degree upward and downward Lorenz dominance clarifies whether focus is turned to changes that take place in the lower or upper part of the income distribution. Thus, a person who favors second-degree upward Lorenz dominance would most likely prefer third-degree upward Lorenz dominance to third-degree downward Lorenz dominance.

As suggested above J_p defined by (3) will be used as basis for judging the normative significance of the criteria of second-degree upward and downward Lorenz dominance. To this end it will be useful to introduce the following notation. Let $\mathbf{P}_2$ be a family of preference functions related to J_P and defined by

$$\mathbf{P}_2 = \left\{ P : P \in \mathbf{P}_1,\ P''' \text{ is continuous on } [0,1] \text{ and } P'''(t) > 0 \text{ for } t \in \langle 0,1 \rangle \right\}.$$

The following result provides a characterization of second-degree upward Lorenz dominance.[10]

THEOREM 2. *Let L_1 and L_2 be members of **L**. Then the following statements are equivalent,*

(i) $$\int_0^u L_1(t)\, dt \geq \int_0^u L_2(t)\, dt \quad \text{for all} \quad u \in [0,1]$$

and the inequality holds strictly for some u

(ii) $$J_P(L_1) < J_P(L_2) \quad \text{for all} \quad P \in \boldsymbol{P}_2,$$

(Proof in Appendix).

[10] Note that a slightly different version of the equivalence between (i) and (ii) in Theorem 2 is proved by Zoli (1999). Actually, when we restrict to cases of equal means Proposition 2 of Zoli (1999) and Theorem 2 yield identical results.

To ensure equivalence between second-degree upward Lorenz dominance and J_P-measures as decision criteria Theorem 2 shows that it is necessary to restrict the preference functions P to be concave with positive third derivatives. If, by contrast, P has negative third derivative, then Theorem 3 yields the analogous to Theorem 2.

Let $\widetilde{\mathbf{P}}_2$ be a family of preference functions related to J_P and defined by

$\widetilde{\mathbf{P}}_2 = \{P : P \in \mathbf{P}_1,\ P''' \text{ is continuous on } [0,1] \text{ and } P'''(t) < 0 \text{ for } t \in \langle 0,1 \rangle\}$.

THEOREM 3. Let L_1 and L_2 be members of **L**. Then the following statements are equivalent,

(i)
$$\int_u^1 \left(1 - L_2(t)\right) dt \geq \int_u^1 \left(1 - L_1(t)\right) dt \quad \text{for all } u \in [0,1]$$

and the inequality holds strictly for some u

(ii)
$$J_P\left(L_1\right) < J_P\left(L_2\right) \quad \text{for all } P \in \widetilde{\boldsymbol{P}}_2,$$

(Proof in Appendix).

The condition of second-degree downward Lorenz dominance proves to be equivalent to the condition that $P'''(t) < 0$ when J_P is used as a ranking criterion for Lorenz curves. A person who employs J_P with $P'''(t) < 0$ considers a given transfer of money from a richer to a poorer person to be more equalizing the higher it occurs in the income distribution provided that the proportions of the population located between the receivers and the donors are equal. As will become evident later it appears convenient to denote this condition the principle of first-degree upside positional transfer sensitivity (first-degree UPTS). Thus, to avoid any confusion we denote Mehran's principle of transfer sensitivity the principle of first-degree downside positional transfer sensitivity (first-degree DPTS).

A person that supports the criterion of second-degree upward Lorenz dominance will assign more weight to changes that take place in the lower part of the Lorenz curve than to changes that occur in the upper part of the Lorenz curve. By contrast, the criterion of second-degree downward Lorenz dominance emphasizes changes that occur in the upper part of the Lorenz curve. Note, however, that both criteria preserve rankings provided by first-degree Lorenz dominance and therefore exhibit inequality aversion.

Theorem 2 and 3 demonstrate that the principles of upward and downward Lorenz dominance can be used to divide J_P-measures into wide families of inequality measures that differ in the measures' sensitivity to changes (transfers) in the lower or upper part of the Lorenz curve.

Members of the family $\{J_P : P \in \mathbf{P}_2\}$ give more weight to changes that take place lower down in the Lorenz curve, while the members of the family $\{J_P : P \in \widetilde{\mathbf{P}}_2\}$ give more weight to changes higher up in the Lorenz curve. Note that $P(t) = 2t - t^2$, the P-function that corresponds to the Gini coefficient, is the only member of $\mathbf{P}_1$ that is neither included in $\mathbf{P}_2$ nor in $\widetilde{\mathbf{P}}_2$. Thus, the Gini coefficient is the only member of the family $\{J_P : P \in \mathbf{P}_1\}$ that neither preserves second-degree upward Lorenz dominance nor second-degree downward Lorenz dominance apart from the case when the inequality in (i) of Theorems 2 and 3 holds strictly for $u = 1$. The suggestion of Muliere and Scarsini (1989) that the Gini coefficient is coherent with second-degree upward Lorenz dominance appears to be in conflict with this result. However, by assuming that the Lorenz curves cross only once the following result may be derived from Theorem 2.[11]

COROLLARY 1. Assume that *L_1 and L_2 are singly intersecting Lorenz curves and let G_1 and G_2 be the two corresponding Gini coefficients. Then the following statements are equivalent,*

(i) $$J_P(L_1) < J_P(L_2) \text{ for all } P \in \boldsymbol{P}_2$$

(ii) $$G_1 \leq G_2$$

and L_1 crosses L_2 initially from above.

(Proof in Appendix)

Note that Corollary 1 is the analogous to the result of Shorrocks and Foster (1987) referred to above.

As a preference ordering on **L**, the Gini coefficient in general favors neither the lower nor the upper part of the Lorenz curves. Therefore, if we restrict the ranking problem to Lorenz curves with equal Gini coefficients, second-degree upward and downward dominance coincide in the sense that a Lorenz curve L_1 that second-degree upward dominates a Lorenz curve L_2 is always second-degree downward dominated by L_2. Thus, it is clear that L_1 (and the corresponding distribution function) can be attained from L_2 (the corresponding distribution function) by employing a set of progressive

[11] Zoli (1999) provides a similar result for singly intersecting generalized Lorenz curves under the condition of equal means.

transfers in combination with an equal set of regressive transfers which leaves the Gini mean difference unchanged and in which the progressive transfers occur in the lower part of L_2. We call such a change *a downside mean-Gini-preserving transformation (downside MGPT)*[12]. Thus, inequality is transferred from lower to higher parts of the Lorenz curve. By contrast, a person who favors second-degree downward Lorenz dominance will apply the progressive transfers in the upper part rather than in the lower part of the Lorenz curve. Such a change will be called *an upside mean-Gini-preserving transformation (upside MGPT).* In this case inequality is transferred from the higher to the lower parts of the Lorenz curve and the corresponding income distribution. Note that the MGPT principles are analogous to the principle of mean-variance-preserving transformation (MVPT) introduced by Menezes et al. (1980). The major difference between these two principles is that the (downside) MGPT principle is equivalent to second-degree upward Lorenz dominance (third-degree upward inverse stochastic dominance) whilst the MVPT principle is equivalent to third degree (upward) stochastic dominance. Moreover, the MGPT principle relies on the Gini mean difference rather than the variance as a measure of spread[13]. When the ranking problem is restricted to Lorenz curves with equal Gini coefficients then the condition for satisfying the principle of downside MGPT (upside MGPT) is equivalent to the conditions for satisfying the principles of transfers and first-degree DPTS (UPTS). The following corollary, which is a direct implication of Theorems 2 and 3, demonstrates that J_P satisfies the principles of downside MGPT (or the principles of transfers and first-degree DPTS) if and only if the corresponding preference functions have negative second and positive third derivatives. By contrast, J_P satisfies the principle of upside MGPT if and only if the preference function has a negative third derivative as well as a negative second derivative.

COROLLARY 2. *Let L_1 and L_2 be Lorenz curves with equal Gini coefficients. Then the following statements are equivalent.*

(i)
$$\int_0^u L_1(t)\,dt \geq \int_0^u L_2(t)\,dt \quad \text{for all } u \in [0,1]$$

(ii)
$$\int_u^1 \left(1 - L_2(t)\right)dt \leq \int_u^1 \left(1 - L_1(t)\right)dt \quad \text{for all } u \in [0,1]$$

where the inequalities in (i) and (ii) hold strictly for some u

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[12] Note that an MGPT is equivalent to the favorable composite positional transfer discussed by Zoli (1997).

(iii) *$J_P(L_1)<J_P(L_2)$ for all J_P that obeys the principles of transfers and first-degree DPTS*

(iv) *$J_P(L_1)>J_P(L_2)$ for all J_P that obeys the principles of transfers and first-degree UPTS*

(v) *L_1 can be attained from L_2 by employing the principle of downside mean-Gini-preserving transformation.*

(vi) *L_2 can be attained from L_1 by employing the principle of upside mean-Gini-preserving transformation.*

3. Lorenz dominance of i^th^ degree

Since situations where second-degree (upward or downward) Lorenz dominance does not provide unambiguous ranking of Lorenz curves may arise it will be useful to introduce weaker dominance criteria than second-degree Lorenz dominance. To this end we will introduce two hierarchical sequences of nested Lorenz dominance criteria; one departs from second-degree upward Lorenz dominance and the other from second-degree downward Lorenz dominance. The choice between second-degree upward and downward Lorenz dominance clarifies whether focus is turned to changes that take place in the lower or upper part of the income distribution. Thus, a person who favors second-degree upward Lorenz dominance would most likely prefer third-degree and higher degrees of upward Lorenz dominance to third-degree and higher degrees of downward Lorenz dominance. Conversely, when the value judgment of a person is consistent with the criterion of second-degree downward Lorenz dominance higher degrees of downward Lorenz dominance is likely more acceptable than higher degrees of upward Lorenz dominance

As will become evident below it is convenient to use the following notation,

(4) $$H_2(u)=\int_0^u L(t)dt,\ \ 0\le u\le 1,$$

$$H_{i+1}(u)=\int_0^u H_i(t)dt,\ \ 0\le u\le 1,\ \ i=2,3,...,$$

and

(5) $$\tilde{H}_2(u)=\int_u^1 \big(1-L(t)\big)dt,\ \ 0\le u\le 1,$$

[13] Note that the Gini mean difference was used as an (robust) alternative to the variance as a measure of spread long before Gini introduced it as a measure of inequality (see David, 1968).

$$\tilde{H}_{i+1}(u) = \int_u^1 \tilde{H}_i(t)dt, \ \ 0 \le u \le 1, \ \ i = 2,3,....$$

Now, using integration by parts, we obtain the following alternative expressions for H_{i+1} and $\tilde{H}_{i+1}$, respectively,

$$H_{i+1}(u) = \frac{1}{(i-1)!} \int_0^u (u-t)^{i-1} L(t)dt, i = 2,3,... \tag{6}$$

and

$$\tilde{H}_{i+1}(u) = \frac{1}{(i-1)!} \int_u^1 (t-u)^{i-1} (1-L(t))dt, i = 2,3,.... \tag{7}$$

Expressions (6) and (7) show that H_{i+1} and $\tilde{H}_{i+1}$ place more weight on changes in the lower and upper part of the Lorenz curve as i increases.

Now, let $P^{(j)}$ denote the j^{th} derivative of P and let $\mathbf{P}_i$ and $\mathbf{P}_i$ be families of preference functions defined by

$$\mathbf{P}_i = \{P : P \in \mathbf{P}_1, P^{(j)} \text{ is continuous on } [0,1], (-1)^{j-1} P^{(j)}(t) > 0 \text{ for } t \in \langle 0,1 \rangle, j = 3,4,\dots,i+1 \\ \text{and } P^{(j)}(1) = 0, \ j = 2,3,\dots,i-1\}$$

and

$$\tilde{\mathbf{P}}_i = \{P : P \in \mathbf{P}_1, P^{(j)} \text{ is continuous on } [0,1], P^{(j)}(t) < 0 \text{ for } t \in \langle 0,1 \rangle, j = 3,4,\dots,i+1 \\ \text{and } P^{(j)}(1) = 0, \ j = 2,3,\dots,i-1\},$$

respectively.

As generalizations of Definitions 2 and 3 we introduce the concepts of i^{th}-degree upward and downward Lorenz dominance[14]. Note that subscripts i and j in the notation $H_{i,j}$ and $\tilde{H}_{i,j}$ used below refer to dominance of i^{th} degree for Lorenz curve L_j and that $H_{1,j}$ is the Lorenz curve L_j.

DEFINITION 4. *A Lorenz curve L_1 is said to i^{th}-degree upward dominate a Lorenz curve L_2 if*

$$H_{i,1}(u) \ge H_{i,2}(u) \ \ \text{for all} \ \ u \in [0,1]$$

[14] A similar definition of i^{th} degree (upward) inverse stochastic dominance was introduced by Muliere and Scarsini (1989). Note that Definitions 4 and 5 do not require any restrictions on the Lorenz curves (or the distribution functions) and thus differ from the definitions of stochastic dominance proposed by Whitmore (1970) and Chew (1983).

and the inequality holds strictly for some u.

DEFINITION 5. *A Lorenz curve L_1 is said to i^{th}-degree downward dominate a Lorenz curve L_2 if*

$$\tilde{H}_{i,2}(u) \geq \tilde{H}_{i,1}(u) \quad \text{for all} \quad u \in [0,1]$$

and the inequality holds strictly for some u.

Note that $(i+1)^{th}$-degree upward and downward Lorenz dominance are less restrictive dominance criteria than i^{th}-degree upward and downward Lorenz dominance and thus can prove to be useful decision criteria in situations where i^{th}-degree dominance does not yield an unambiguous ranking of Lorenz curves.

It follows from the definitions (6) and (7) of H and $\tilde{H}$ that

$$H_{i,1}(u) \geq H_{i,2}(u) \quad \text{for all} \quad u$$

implies

$$H_{i+1,1}(u) \geq H_{i+1,2}(u) \quad \text{for all} \quad u,$$

and that

$$\tilde{H}_{i,2}(u) \geq \tilde{H}_{i,1}(u) \quad \text{for all} \quad u$$

implies

$$\tilde{H}_{i+1,2}(u) \geq \tilde{H}_{i+1,1}(u) \quad \text{for all} \quad u.$$

Thus, the various degrees of upward and downward Lorenz dominance form two separate sequences of nested dominance criteria, which turn out to be useful for dividing the J_P-family of inequality measures into nested subfamilies. The subfamilies of the J_P-family are characterized by the following theorems.

THEOREM 4. *Let L_1 and L_2 be members of* $\boldsymbol{L}$. *Then*

$$H_{i,1}(u) \geq H_{i,2}(u) \quad \text{for all} \quad u \in [0,1]$$

and the inequality holds strictly for some u if and only if

$$J_P(L_1) < J_P(L_2) \quad \text{for all} \quad P \in \boldsymbol{P}_i.$$

(Proof in Appendix).

THEOREM 5. *Let L_1 and L_2 be members of **L**. Then*

$$\tilde{H}_{i,2}(u) \geq \tilde{H}_{i,1}(u) \quad \text{for all} \quad u \in [0,1]$$

and the inequality holds strictly for some u if and only if

$$J_P(L_1) < J_P(L_2) \quad \text{for all} \quad P \in \tilde{\boldsymbol{P}}_i \, .$$

(Proof in Appendix).

To judge the normative significance of i^{th} degree upward and downward Lorenz dominance and the restrictions these principles impose on the preference functions (weight function) of the J_P-family of inequality measures, it appears helpful to strengthen the principles of first-degree downside and upside positional transfer sensitivity to be more sensitive to transfers that take place lower down (higher up) in the income distribution. To this end the first stage will be to introduce the following definition (see the proof of Corollary 3 in the Appendix for a more formal definition).

DEFINITION 6. *When a sequence of first-degree DPTS (UPTS) transfers are valued more the lower down (higher up) the transfers occur the sequence of transfers are said to be made in line with the principle of second-degree downside (upside) positional transfer sensitivity. We denote this principle second-degree DPTS (UPTS).*

As we demonstrated in Section 2 (Corollary 2) first-degree DPTS (UPTS) could be given an alternative interpretation in terms of a mean-Gini-preserving transformation. This equivalence arises due to the fact that second-degree upward and downward Lorenz dominance "coincide" when the Gini coefficient (or the mean and the Gini mean difference) of the Lorenz curves are equal. By assuming that $H_3(1)$ defined by (6) is kept fixed a similar interpretation of second-degree DPTS in terms of a mean-$H_3(1)$-preserving transformation can be obtained. It is easily verified that $H_{i+1}(1)$ defined by (6) simply is a linear transformation of a measure of inequality that belongs to the extended Gini family of inequality measures[15],

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[15] The extended Gini family of inequality measures was introduced by Donaldson and Weymark (1980, 1983) and Kakwani (1980).

(8) $$H_{i+1}(1) = \frac{1}{(i+1)!}\left(1 - G_i(L)\right), \; i = 1, 2, ...$$

where

(9) $$G_i(L) = 1 - i(i+1)\int_0^1 (1-u)^{i-1} L(u)\,du, \; i \geq 1.$$

Thus, requiring $H_3(1)$ to be equal across Lorenz curves is equivalent to require that the extended Gini measure of inequality $G_2(L)$ (or its absolute version and the mean) is kept fixed. Therefore, if we restrict the ranking problem to Lorenz curves with equal G_2-coefficients second-degree upward and downward dominance of $H_2(u)$ "coincide". Observe that second-degree upward dominance of $H_2(u)$ is equivalent to third-degree upward Lorenz dominance whilst downward dominance of $H_2(u)$ differs from third-degree downward Lorenz dominance. Accordingly, we have that G_2 plays a similar role for third-degree upward Lorenz dominance as the Gini coefficient (G_1) plays for second-degree upward (and downward) Lorenz dominance. As for second-degree upward Lorenz dominance L_1 can be attained from L_2 by employing a set of progressive transfers in combination with an equal set of regressive transfers which leaves the mean and the G_2-coefficient unchanged. A person who supports third-degree upward Lorenz dominance will apply the progressive transfers in the lower part of the Lorenz curve.

As indicated above there is a similarity between third-degree and second-degree Lorenz dominance in the sense that both dominance criteria can be given a normative characterization in terms of principles for positional transfers sensitivity and mean-"spread"-preserving transformations. However, in contrast to the second-degree dominance criteria downward and upward third-degree Lorenz dominance require different measures of "spread" (inequality) to be kept fixed in order to obey the principle of mean-"spread"-preserving transformation[16]. This follows from the fact that $H_i(1)$ and $\tilde{H}_i(0)$ represent different measures of inequality when $i > 2$. From the definition (7) of $\tilde{H}$ we get that

(10) $$\tilde{H}_{i+1}(0) = \frac{1}{(i+1)!}\left(i\,D_i(L) + 1\right), \; i = 1, 2, ...$$

where

[16] In contrast to the absolute Gini-coefficient note that the absolute versions of G_i defined by (9) and D_i defined by (11) do not fulfill the standard conditions of being measures of spread (dispersion) when i>1.

(11) $$D_i(L) = 1 - (i+1)\int_0^1 u^{i-1}L(u)du,\ i = 1,2,...$$

is an alternative "generalized" Gini family of inequality measures denoted the Lorenz family of inequality measures[17]. As demonstrated by Aaberge (2000a) the integer subscript subclass of the extended Gini family and the Lorenz family of inequality measures are equivalent in the sense that any (integer subscript) extended Gini measure is uniquely determined by the Lorenz measures of inequality and vice versa. The essential difference between these measures is that the extended Gini measures focus attention on changes that concern the lower part of the Lorenz curves whilst the Lorenz measures are particularly sensitive to changes that take place in the upper part of the Lorenz curves. This property explains why a person who supports third-degree downward Lorenz dominance will apply the progressive transfer in the lower part of the Lorenz curve while a person who supports third-degree downward Lorenz dominance will apply the progressive transfer in the upper part of the Lorenz curve. These results are summarized in Corollaries 3 and 4 which, however, deal with i^{th}-degree Lorenz dominance rather than third-degree Lorenz dominance. Thus, definitions of i^{th} degree DPTS (UPTS), mean-G_i-preserving transformation (MG_iPT) and mean-D_i-preserving transformation (MD_iPT) are required.

DEFINITION 7. *When a sequence of* $(i\text{-}1)^{th}$*- degree DPTS (UPTS) transfers are valued more the lower down (higher up) the transfers occur the sequence of transfers are said to be made in line with the principle of i -degree downside (upside) positional transfer sensitivity. We denote this principle* i^{th}*-degree DPTS (UPTS).*

DEFINITION 8. *A mean-*G_i*-preserving transformation (*MG_iPT*) is a combination of a progressive and regressive transfer that leaves the mean and the* G_i*-coefficient unchanged and where the progressive transfer occurs in the lower part of the income distribution.*

DEFINITION 9. *A mean-*D_i*-preserving transformation (*MD_iPT*) is a combination of a progressive and regressive transfer that leaves the mean and the* D_i *-coefficient unchanged and where the progressive transfer occurs in the upper part of the income distribution.*

COROLLARY 3. *Let* L_1 *and* L_2 *be Lorenz curves with equal* G_i*-coefficients (defined by (9). Then the following statements are equivalent*

(i) $H_{i+1,1}(u) \geq H_{i+1,2}$ *for all* $u \in [0,1]$ *and the inequality holds strictly for some u,*

(ii) $J_P(L_1)<J_P(L_2)$ *for all* J_P *that obeys the principle of transfers and the principles of DPTS up to and including* i^{th} *degree*

(iii) L_1 *can be attained from* L_2 *by employing the* MG_iPT *principle.*

A formal proof of the equivalence between (i) and (ii) for third-degree upward Lorenz dominance in Corollary 3 is given in the Appendix. A formal proof of the equivalence between (i) and (iii) requires considerable additional notation and space and is thus omitted. However, note that this proof is analogous to the proof of the equivalence between third-degree stochastic dominance and the principle of mean-variance-preserving transformation provided by Menezes et al. (1980).

COROLLARY 4. *Let* L_1 *and* L_2 *be Lorenz curves with equal* D_i*-coefficients (defined by (11). Then the following statements are equivalent*

(i) $\tilde{H}_{i+1,2} \geq \tilde{H}_{i+1,1}$ *for all* $u\in[0,1]$ *and the inequality holds strictly for some u,*

(ii) $J_P(L_1)<J_P(L_2)$ *for all* J_P *that obeys the principle of transfers and the principles of UPTS up to and including* i^{th} *degree*

(iii) L_1 *can be attained from* L_2 *by employing the* MD_iPT *principle.*

The proposed sequences of dominance criteria along with the results of Theorems 4 and 5 and Corollaries 3 and 4 suggest two alternative strategies for increasing the number of Lorenz curves that can be strictly ordered by successively narrowing the class of inequality measures under consideration. As the dominance criteria of each sequence are nested these strategies also allow us to identify the value judgments that are needed to reach an unambiguous ranking of Lorenz curves. It follows from Theorem 4 that J_P-measures derived from P-functions with derivatives between second and i^{th} order that alternate in sign $\left((-1)^{j-1}\, P^{(j)}(t)>0,\, j=2,3,\ldots,i\right)$ preserve upward Lorenz dominance of all degrees lower than and equal to i-1. Thus, as demonstrated by Corollary 3 their sensitivity to changes that occur in the lower part of the income distribution (and the Lorenz curve) increases as i increases. By contrast, Theorem 5 shows that J_P-measures derived from P-functions with negative derivatives of order two and up to i $\left(P^{(j)}(t)<0,\, j=2,3,\ldots,i\right)$ preserve downward Lorenz dominance of all degrees lower than and equal to i-1. Corollary 4 demonstrates that this means that they increase their sensitivity to changes that occur in the upper part of the Lorenz curve as i increases. Note that Theorems 4 and 5 are only valid for finite i. At the extreme, as $i\rightarrow\infty$, observe that

[17] The Lorenz family of inequality measures was introduced by Aaberge (2000a) and proves to be a subclass of

(12) $$\left(i+1\right)!H_{i+1}(u) \to \begin{cases} 0, & 0 \le u < 1 \\ \dfrac{F^{-1}(0+)}{\mu}, & u = 1 \end{cases}$$

and

(13) $$\left(i+1\right)!\tilde{H}_{i+1}(u) \to \begin{cases} \dfrac{F^{-1}(1)}{\mu}, & u = 0 \\ 0, & 0 < u \le 1, \end{cases}$$

where $F^{-1}(0+)$ and $F^{-1}(1)$ denote the lowest and highest income, respectively. Hence, at the limit upward and downward Lorenz dominance solely depend on the income share of the worst-off and best-off income recipient, respectively. At the extreme upward Lorenz dominance is solely concerned with transfers that benefit the poorest unit. By contrast, downward Lorenz dominance solely focuses on transferring money from the richest to anyone else.

REMARK. Restricting the comparisons of Lorenz curves to distributions with equal means the dominance results of Theorems 1-5 and Corollaries 1-4 are valid for generalized Lorenz curves and also apply to the so-called dual theory representation for choice under uncertainty introduced by Yaari (1987, 1988).

4. The relationship between downward and upward Lorenz dominance and generalized Gini families of inequality measures

The dominance results in Sections 2 and 3 show that application of the criteria of upward Lorenz dominance requires a higher degree of aversion to downside inequality the higher is the degree of upward Lorenz dominance. A similar relationship holds between downward Lorenz dominance and aversion to upside inequality aversion. As suggested in Section 3 the highest degree of downside inequality aversion is achieved when focus is exclusively turned to the situation of the worst-off income recipient. Thus, the most downside inequality averse J_P-measure is obtained as the preference function approaches

(14) $$P_d(t) = \begin{cases} 0, & t = 0 \\ 1, & 0 < t \le 1. \end{cases}$$

As P_d is not differentiable, it is not a member of the family $\mathbf{P}_1$ of inequality averse preference functions, but it is recognizable as the upper limit of inequality aversion for members of $\mathbf{P}_1$. Inserting (14) in (3) yields

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the "illfare-ranked single-series Ginis" discussed by Donaldson and Weymark (1980) and Bossert (1990).

(15) $$J_{P_d}(L) = 1 - \frac{F^{-1}(0+)}{\mu}.$$

Hence, the inequality measure J_{P_d} corresponds to the Rawlsian maximin criterion. Since J_{P_d} is compatible with the limiting case of upward Lorenz dominance the Rawlsian (relative) maximin criterion preserves all degrees of upward Lorenz dominance and rejects downward Lorenz dominance.

By contrast, the most upside inequality averse J_P-measure is obtained as P approaches[18]

(16) $$P_u(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & t = 1. \end{cases}$$

Inserting (16) in (3) yields

(17) $$J_{P_u}(L) = 1 + \frac{F^{-1}(1)}{\mu}.$$

Thus, J_{P_u}, which we will denote the relative minimax criterion, is "dual" to the Rawlsian (relative) maximin criterion in the sense that it is compatible with the limiting case of downward Lorenz dominance. When the comparison of Lorenz curves is based on the relative minimax criterion the Lorenz curve for which the largest relative income is smaller is preferred, regardless of all other differences. The only transfer which decrease inequality is a transfer from the richest unit to anyone else.

Based on the results of Theorems 1-5 and Corollaries 1-4, we shall now demonstrate how the above Lorenz dominance results can be applied to evaluate the ranking properties of the Lorenz and extended Gini families of inequality measures. The extended Gini family is defined by (9). Note that $\{G_k : k > 0\}$ is a subfamily of $\{J_P : P \in \mathbf{P}_1\}$ formed by the following family of P-functions,

(18) $$P_k(t) = 1 - (1-t)^{k+1}, k \ge 0.$$

Differentiating P_k defined by (16), we find that

(19) $$P_k^{(j)}(t) = \begin{cases} (-1)^{j-1} \dfrac{(k+1)!}{(k-j+1)!}(1-t)^{k-j+1}, & j = 1,2,\dots,k+1 \\ 0 & , \quad j = k+2, k+3,\dots \end{cases}$$

Equation (19) implies that $P_k''(t) < 0$ for all $t \in \langle 0,1 \rangle$ when $k > 0$ and thus that the G_k-measures satisfy the principle of transfers for $k > 0$. Moreover, $P_k'''(t) > 0$ for all $t \in \langle 0,1 \rangle$ when $k > 1$. Hence all G_k for $k > 1$ preserve second-degree upward Lorenz dominance. Moreover, the derivatives of P_k alternate in

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[18] Note that the normalization condition P'(0)=1 is ignored in this case.

sign up to the $(k+1)^{th}$ derivative and $P_k^{(j)}(1)=0$ for all $j \le k$. Thus, it follows from Theorem 4 that G_k preserves upward Lorenz dominance of degree k and therefore also preserves upward Lorenz dominance for all degrees lower than k. Finally, it follows from Theorem 4 and Corollary 3 that G_{k+1} exhibits stronger downside inequality aversion than G_k for $k>0$. Therefore, within the family $\{G_k : k=1,2,...\}$ of inequality measures increasing the degree of downside inequality aversion corresponds to support an increase in the degree of upward Lorenz dominance. Hence, the cost of inequality is higher when measured by G_{k+1} than by G_k. The most inequality averse behavior occurs as $k \to \infty$, which corresponds to the inequality averse behavior of the Rawlsian (relative) maximin criterion. Thus, G_k satisfies all degrees of upward Lorenz dominance as $k \to \infty$. At the other extreme, as $k=0$, the preference function P_0 exhibits inequality neutrality. Note that G_k for $0<k<1$ preserves downward Lorenz dominance of all degrees. The stated properties of the G_k-measures are summarized in the following corollary,

COROLLARY 5. *The extended Gini family of inequality measures defined by (9) has the following properties,*

(i) *G_k preserves upward Lorenz dominance of degree k and all degrees lower than k,*

(ii) *G_k satisfies the principle of transfers for $k>0$,*

(iii) *G_k satisfies the principle of transfers and the principles of DPTS up to and including $(k\text{-}1)^{th}$ degree,*

(iv) *G_k satisfies the principle of mean-G_i-preserving transformation for i=1, 2,...,k-1,*

(v) *G_k exhibits more downside inequality aversion than G_{k-1},*

(vi) *G_k approaches inequality neutrality as $k \to 0$.*

(vii) *G_k approaches the Rawlsian relative maximin criterion as $k \to \infty$.*

As demonstrated by Aaberge (2000a) the Lorenz family of inequality measures is a subfamily of $\{J_P : P \in \mathbf{P}_1\}$ formed by the following family of P-functions,

(20)
$$P_k(t) = \frac{1}{k}\left((k+1)t - t^{k+1}\right), k=1,2,...$$

Differentiating P_k defined by (20) yields

(21)
$$P_k^{(i)}(t) = \begin{cases} -(k+1)(k-1)(k-2)\,(k-i+2)t^{k-i+2}, & i=1,2,...,k+1 \\ 0 & , i=k+2, k+3 \end{cases}$$

The results of a similar evaluation of the Lorenz family of inequality measures as that carried out for the extended Gini family are summarized in the following corollary.

COROLLARY 6. *The Lorenz family of inequality measures defined by (11) has the following properties,*

(i) D_k *preserves downward Lorenz dominance of degree k and all degrees lower than k,*

(ii) D_k *satisfies the principle of transfers for all k,*

(iii) D_k *satisfies the principle of transfers and the principles of UPTS up to and including (k-1)*th *degree,*

(iv) D_k *satisfies the principle of mean-*D_i*-preserving transformation for i=1, 2,...,k-1,*

(v) D_k *exhibits more upside inequality aversion than* D_{k-1}*,*

(vi) D_k *approaches inequality neutrality as* $k \to \infty$.

(vii) $k\,D_k + 1$ *approaches the relative minimax criterion as* $k \to \infty$.

Note that D_k approaches the Rawlsian relative maximin as k approaches -1.

Aaberge (2000a) demonstrated that the Lorenz family of inequality measures as well as the integer subscript subfamily of the extended Gini family uniquely determines the Lorenz curve, i.e. examination of inequality in an income distribution can, without loss of information, either be restricted to the Lorenz family or the family $\{G_k : k = 1, 2, ...\}$ of inequality measures. Thus, combining this result with the results of Theorems 4 and 5 yield[19]

THEOREM 6. *Let* L_1 *and* L_2 *be members of* ***L***. *Then the following statements are equivalent*

(i)
$$H_{i1}(u) \geq H_{i2}(u) \ \ for \ \ all \ \ u \in [0,1]$$

and the inequality holds strictly for some u

(ii)
$$G_k(L_1) < G_k(L_2) \ \ for \ \ k = i, i+1, i+2, ...$$

THEOREM 7. *Let* L_1 *and* L_2 *be members of* ***L***. *Then the following statements are equivalent* *(i)* $$\tilde{H}_{i,2}(u) \geq \tilde{H}_{i,1}(u) \;\; \text{for all} \;\; u \in [0,1]$$

and the inequality holds strictly for some u

(ii) $$D_k\left(L_1\right) < D_k\left(L_2\right) \;\; \text{for} \;\; k = i, i+1, i+2, \ldots .$$

and $$\frac{F_1^{-1}(1)}{\mu_1} < \frac{F_2^{-1}(1)}{\mu_2}.$$

Theorem 6 shows that characterizations of the various degrees of upward Lorenz dominance can be made in terms of conditions for the extended Gini measures of inequality which divide the integer subscript subclass of the extended Gini family into nested subfamilies. Thus, the hierarchical sequence of nested upward Lorenz dominance criteria offers a convenient computational method for identifying the largest subfamily of the integer subscript extended Gini family of inequality measures that yields an unambiguous ranking of Lorenz curves. As demonstrated by Theorem 7 the various degrees of downward Lorenz dominance divide the Lorenz family of inequality measures into a similar sequence of nested subfamilies. This means that we are able to identify the restrictions on the preference functions and therefore the least restrictive value judgment required reaching an unambiguous conclusion irrespective of whether we are averse to downside or upside inequality. An interesting common feature of the two alternative strategies for ranking Lorenz curves provided by Theorems 6 and 7 is that both strategies depart from the Gini coefficient; one of them requires higher and the other lower degree of downside inequality aversion than what is exhibited by the Gini coefficient.

5. Summary and discussion

This paper introduces two sequences of partial orderings for achieving complete rankings of Lorenz curves. In particular, we have examined situations where Lorenz curves intersect by introducing ranking criteria that are weaker than non-intersecting dominance (first-degree Lorenz dominance) and stronger than single measures of inequality. The proposed dominance criteria are shown to characterize nested subsets of the families of inequality measures defined by $\int P'(u) dL(u)$ where P' is the derivatives of a function that defines the inequality aversion profile of the inequality measure. The condition of first-degree Lorenz dominance corresponds to concave P-functions. By introducing higher degrees of dominance, this paper provides a method for identifying the lowest degree of dominance and the weakest restriction on the functional form of the preference functions P that is

[19] Muliere and Scarsini (1979) proved the sufficiency part of Theorem 6 ((i) implies (ii)) and provided a characterization of i^{th} degree upward Lorenz dominance (inverse stochastic dominance) in terms of order

needed to reach unambiguous rankings of Lorenz curves irrespective of whether one's social preferences is consistent with downside or upside inequality aversion. To judge the normative significance of the sequences of dominance criteria appropriate principles of transfers and mean-"spread"-preserving transformations has been introduced. The criteria of Lorenz dominance provide convenient computational methods. Thus, in applied work the ranking obtained by applying this approach should in general have a wider degree of support than that obtained by applying summary measures of inequality.

To deal with the mean income income inequality trade-off, in cases where they conflict, Shorrocks (1983) introduced the "generalized Lorenz curve", defined as a mean scaled-up version of the Lorenz curve. Moreover, Shorrocks (1983) obtained characterizations of social welfare functions based on first-degree dominance relations between generalized Lorenz curves. Thus, scaling up the introduced Lorenz dominance relations of this paper by the mean income (μ) it can be demonstrated that the present results also apply to the generalized Lorenz curve and moreover provide convenient characterizations of the corresponding social welfare orderings.

Appendix

Proofs of Dominance Results

LEMMA 1. *Let H be the family of bounded, continuous and non-negative functions on [0,1] which are positive on* $\langle 0,1\rangle$ *and let g be an arbitrary bounded and continuous function on [0,1]. Then*

$$\int g(t)\,h(t)\,dt > 0 \quad \text{for all} \quad h \in H$$

implies

$$g(t) \geq 0 \quad \text{for all} \quad t \in [0,1]$$

and the inequality holds strictly for at least one $t \in \langle 0,1\rangle$.

The proof of Lemma 1 is known from mathematical textbooks.

The proofs of Theorem 1 and 2 follow from Hadar and Russel (1969) but are included below for the sake of completeness.

Proof of Theorem 1. From the definition (2) of $J_P(L)$ it follows that

conditions for a (large) sub-family of $\{J_P : P \in \mathbf{P}_i\}$.

$$J_P(L_2) - J_P(L_1) = -\int_0^1 P''(u)\big(L_1(u) - L_2(u)\big)du .$$

Thus, if (i) holds, then $J_P(L_2) - J_P(L_1) > 0$ for all $P \in \mathbf{P}_1$.

Conversely, by assuming that (ii) is true, application of Lemma 1 gives (i). Hence, the equivalence of (i) and (ii) is proved.

Q.E.D.

Proof of Theorem 2. Using integration by parts we have that

$$J_P(L_2) - J_P(L_1) = -P''(1)\int_0^1 \big(L_1(u) - L_2(u)\big)du + \int_0^1 P'''(u)\int_0^u \big(L_1(t) - L_2(t)\big)dt\,du .$$

Thus, if (i) holds then $J_P(L_2) > J_P(L_1)$ for all $P \in \mathbf{P}_2$.

To prove the converse statement we restrict to preference functions $P \in \mathbf{P}_2$ for which $P''(1) = 0$. Hence,

$$J_P(L_2) - J_P(L_1) = \int_0^1 P'''(u)\int_0^u \big(L_1(t) - L_2(t)\big)dt\,du$$

and the desired result it obtained by applying Lemma 1.

Q.E.D.

Proof of Corollary 2. The statement (i) implies (ii) follows from Theorem 2.

To prove the converse statement assume that (ii) holds and that L_1 and L_2 cross at u=a. Then it follows that the inequalities

(I) $$\int_0^a \big(L_1(u) - L_2(u)\big)du > 0$$

and

(II) $$\int_0^1 \big(L_1(u) - L_2(u)\big)du = \frac{1}{2}\big(G_2 - G_1\big) \geq 0 .$$

Since L_1 and L_2 cross only once (1) and (2) implies that

(III) $$\int_0^u \left(L_1(u)-L_2(u)\right)du \ge 0 \text{ for all } u\in\left[0,1\right]$$

and the inequality holds strictly for some u, and the desired result is obtained by applying Theorem 2.

Q.E.D

The proof of Theorem 3 is analogous to the proof of Theorem 2 and is based on the expression

$$J_P\left(L_2\right)-J_P\left(L_1\right)=-P''(0)\int_0^1 \left(L_1(t)-L_2(t)\right)dt-P'''(u)\int_u^1 \left(L_1(t)-L_2(t)\right)dt\,du$$

which is obtained by using integration by parts. Thus, by arguments like those in the proof of Theorem 2 the results of Theorem 3 is obtained.

Proof of Theorem 4. To examine the case of i[th] degree upward Lorenz dominance we integrate $J_P\left(L_2\right)-J_P\left(L_1\right)$ by parts i times,

$$J_P\left(L_2\right)-J_P\left(L_1\right)=\sum_{j=2}^{i} (-1)^{j-1}P^{(j)}(1)\left(H_{j,1}(1)-H_{j,2}(1)\right)+(-1)^i\int_0^1 P^{(i+1)}(u)\left(H_{i,1}(u)-H_{i,2}(u)\right)du$$

and use this expression in constructing the proof.

Assume first that

$$H_{i,1}(u)-H_{i,2}(u)\ge 0 \text{ for all } u\in\left[0,1\right]$$

and > holds for at least one u.

Then $J_P\left(L_2\right)>J_P\left(L_1\right)$ for all $P\in\mathbf{P}_i$.

Conversely, assume that

$$J_P\left(L_2\right)>J_P\left(L_1\right) \text{ for all } P\in\mathbf{P}_i .$$

Then this statement holds for the subfamily of $\mathbf{P}_i$ for which $P^{(i)}(1)=0$. For this particular family of preference functions we have that

$$J_P\left(L_2\right)-J_P\left(L_1\right)=(-1)^i\int_0^1 P^{(i+1)}(u)\left(H_{i,1}(u)-H_{i,2}(u)\right)du.$$

Then, as demonstrated by Lemma 1, the desired result can be obtained by a suitable choice of $P\in\mathbf{P}_i$ for which $P^{(i)}(1)=0$.

Q.E.D.

The proofs of Theorem 5, 6 and 7 can be constructed by following exactly the line of reasoning used in the proof of Theorem 2. The proofs use the following expressions,

$$J_P(L_2) - J_p(L_1) = -\sum_{j=2}^{i} P^{(j)}(0)\left(\tilde{H}_{j,2}(0) - \tilde{H}_{j,1}(0)\right) - \int_0^1 P^{(i+1)}(u)\left(\tilde{H}_{i,2}(u) - \tilde{H}_{i,1}(u)\right)du,$$

which is obtained by using integration by parts i times.

Proof of Corollary 3. The equivalence between (i) and (ii).

Inserting for (1) in (3) yields the following alternative expression for J_P in terms of the income distribution F,

$$J_P(F) = 1 - \frac{1}{\mu}\int_0^1 F^{-1}(t)P'(t)\,dt.$$

Now, let us consider a case where we transfer an infinitesimal amount of money from persons with incomes $F^{-1}(t_1+h_1)$, $F^{-1}(t_1+h_1+h_2)$, $F^{-1}(t_2+h_1)$ and $F^{-1}(t_2+h_1+h_2)$ to persons with incomes $F^{-1}(t_1)$, $F^{-1}(t_1+h_2)$, $F^{-1}(t_2)$ and $F^{-1}(t_2+h_2)$, respectively, where t_2 is assumed to be larger than t_1. Moreover, we assume that the transfers are made in line with principle of downside (or upside) positional transfer sensitivity. Thus, the transfers from richer to poorer persons that obey FDPTS (FUPTS) are valued more the lower down (higher up) the transfers occurs if and only if

$$P'(t_1) - P'(t_1+h_1) - \left(P'(t_1+h_2) - P'(t_1+h_1+h_2)\right) > P'(t_2) - P'(t_2+h_1) - \left(P'(t_2+h_2) - P'(t_2+h_1+h_2)\right)$$

which for small h_1 is equivalent to

$$-P''(t_1) + P''(t_1+h_2) > -P''(t_2) + P''(t_2+h_2).$$

For small h_2 this is equivalent to

$$P'''(t_1+h_3) - P'''(t_1) < 0$$

which for small h_3 is equivalent to

$$P''''(t_1) < 0$$

and the proof is complete.

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